Volume in $\mathbb{R}^n$ and norms of ideals

Assume that $S \subset \mathbb{R}^n$ is a bounded open subset

Def The volume of S is

$$Vol(S) = \int_S dx_1 \dots dx_n$$

Rem $Vol(S)$ depends on the choice of coordinates in $\mathbb{R}^n$. Under a linear change of coordinates given by a matrix A, i.e. $x_i = \sum a_{ij} y_j$, $A = (a_{ij})$, the volume is multiplied by $|\det A|$

Def If $\Lambda \subset \mathbb{R}^n$ is a lattice let $Vol(\Lambda) = Vol(\mathbb{R}^n/\Lambda)$

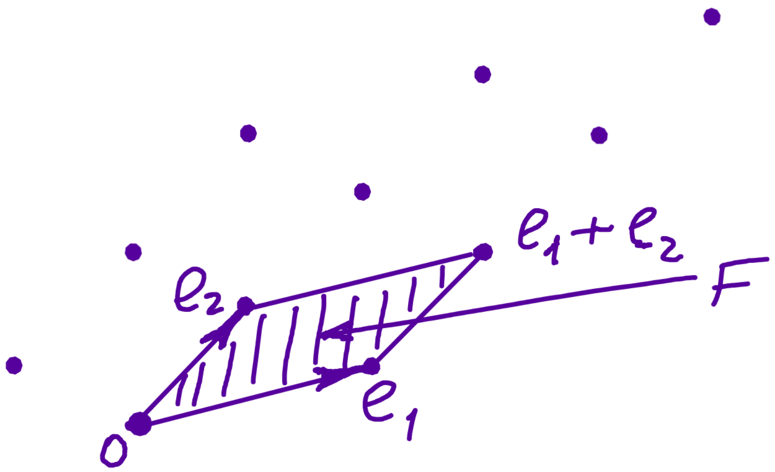
$\mathbb{R}^n/\Lambda$ is compact torus

Rem If $\Lambda = \langle e_1, \dots, e_n \rangle$ define

$$F = \left\{ \sum_{i=1}^n t_i e_i \;\middle|\; t_i \in \mathbb{R},\ 0 < t_i < 1,\ i = 1..n \right\}$$

$F \subset \mathbb{R}^n$ $\qquad$ $\overline{F}$ is a fundamental domain for Λ

$$\mathrm{Vol}(\Lambda) = \mathrm{Vol}(\mathbb{R}^n/\Lambda) = \underset{\displaystyle \int_F dx_1 \ldots dx_n}{\underset{\shortparallel}{\mathrm{Vol}(F)}}$$



Lemma $\Lambda_1 \subset \Lambda_2 \subset \mathbb{R}^n$ two lattices

Then $[\Lambda_2 : \Lambda_1] = |\Lambda_2/\Lambda_1| = \dfrac{\mathrm{Vol}(\Lambda_1)}{\mathrm{Vol}(\Lambda_2)}$

Proof Note that $\dfrac{\mathrm{Vol}(\Lambda_1)}{\mathrm{Vol}(\Lambda_2)}$ does not change under linear change of coordinates in $\mathbb{R}^n \Rightarrow$ we can choose arbitrary basis in $\mathbb{R}^n$ to compute Vol

$\Lambda_2 = \langle e_1, \ldots, e_n \rangle$ $\quad$ $\Lambda_1 = \langle d_1 e_1, \ldots d_n e_n \rangle$

for some e_i, $d_i \in \mathbb{Z}$

In the coord. defined by

the basis $e_1, \dots, e_n$

$$\mathrm{Vol}(\Lambda_2) = \int_{F_2} dx_1 \dots dx_n \,, \quad F_2 = (0,1)^n$$

$$= 1$$

$$\mathrm{Vol}(\Lambda_1) = \int_{F_1} dx_1 \dots dx_n \,, \quad F_1 = (0, d_1) \times \dots \times (0, d_n)$$

$$= \prod_{i=1}^{n} d_i$$

$$\Rightarrow \frac{\mathrm{Vol}(\Lambda_1)}{\mathrm{Vol}(\Lambda_2)} = \prod_{i=1}^{n} d_i = |\Lambda_2 / \Lambda_1| \quad \square$$

<u>Prop 10.3</u> Let $I \in \mathcal{J}(K)$, then

$$\mathrm{Norm}(I) = \frac{\mathrm{Vol}(I)}{\mathrm{Vol}(\mathcal{O}_K)} \,,$$ where

I and $\mathcal{O}_K$ are considered as lattices in $\mathbb{R}^n = K \otimes_{\mathbb{Q}} \mathbb{R}$

<u>Proof</u> If $I \subset \mathcal{O}_K$ an ideal, then

$$\mathrm{Norm}(I) \overset{\text{Prop 10.1}}{=} |\mathcal{O}_K / I| \overset{\text{Lemma}}{=}$$

$$= \frac{\mathrm{Vol}(I)}{\mathrm{Vol}(\mathcal{O}_K)}$$

In general, $I = x \cdot \mathfrak{a}$, for some $x \in K^\times$, $\mathfrak{a} \subset \mathcal{O}_K$ an ideal

$$\mathrm{Norm}(I) = \mathrm{Norm}(x \cdot \mathfrak{a}) \overset{\text{Prop. 10.1}}{=} |N_{K/\mathbb{Q}}(x)| \cdot \mathrm{Norm}(\mathfrak{a}) = \frac{|N_{K/\mathbb{Q}}(x)| \cdot \mathrm{Vol}(\mathfrak{a})}{\mathrm{Vol}(\mathcal{O}_K)}$$

$$\mathrm{Vol}(I) = |\det M_x| \, \mathrm{Vol}(\mathfrak{a})$$

where M_x is multiplication by x, because $I = x \cdot \mathfrak{a}$ is the image of $\mathfrak{a}$ under the linear map M_x

by definition, $N_{K/\mathbb{Q}}(x) = \det M_x$

$$\Rightarrow \mathrm{Norm}(I) = \frac{\mathrm{Vol}(I)}{\mathrm{Vol}(\mathcal{O}_K)} \qquad \square$$