13. Localization and discrete valuation rings

R - a ring

Def A subset $S \subset R$ is called multiplicatively closed, if $1 \in S$ and $a, b \in S \Rightarrow ab \in S$

Assume S is mult. closed subset of R

We want to divide by the elements of S, i.e. we want to make $x \in S$ invertible

Def The localization of R w.r.t. S (or the ring of fractions) denoted $S^{-1}R$ is a ring defined as follows:

consider the set of pairs $(a, t) \in R \times S$ and the equiv. relation

(*) $(a_1, t_1) \sim (a_2, t_2) \Leftrightarrow \exists t_3 \in S : t_3(a_1 t_2 - a_2 t_1) = 0$

The elements of $S^{-1}R$ are equiv. classes denoted by $\frac{a}{t}$, addition, multiplication are defined as usual:

$$\frac{a_1}{t_1} + \frac{a_2}{t_2} = \frac{a_1 t_2 + a_2 t_1}{t_1 t_2}$$

$$\frac{a_1}{t_1} \cdot \frac{a_2}{t_2} = \frac{a_1 a_2}{t_1 t_2}$$

The equiv. relation (*) implies that $\forall \frac{a}{t} \in S^{-1}R$, $s \in S$: $\frac{a}{t} = \frac{as}{ts}$

Exercise: check that this really defines ring str-re on $S^{-1}R$

Rem 1) equiv. relation (*) means:

$$\frac{a_1}{t_1} = \frac{a_2}{t_2} \quad \text{if} \quad \exists t_3 \in S:$$

$$a_1 t_2 \cdot t_3 = a_2 t_1 \cdot t_3$$

t_3 is needed, because otherwise we do not get equiv. relation

2) If R is an integral domain, then (*) can be replaced by:

$$\frac{a_1}{t_1} = \frac{a_2}{t_2} \iff a_1 t_2 = a_2 t_1$$

We have a natural ring homomorphism:

$$\lambda: R \longrightarrow S^{-1}R$$
$$a \longmapsto \frac{a}{1} = a$$

Lemma 13.1 Assume that $\varphi: R_1 \to R_2$ is a ring morphism, $S \subset R_1$ a mult. closed subset, s.t. $\varphi(S) \subset R_2^\times$ (i.e. all elements in $\varphi(S)$ are invertible). Then the map φ factors throug the localization:

$$\begin{tikzcd} R_1 \arrow[r, "\lambda"] \arrow[dr, "\varphi"'] & S^{-1}R_1 \arrow[d, dashed, "\psi"] \\ & R_2 \end{tikzcd}$$

$\exists$ unique $\psi: S^{-1}R_1 \to R_2$, s.t. $\varphi = \psi \circ \lambda$

<u>Proof</u> Define: $\psi\left(\frac{a}{t}\right) = \varphi(a)\cdot\varphi(t)^{-1}$

(since $\varphi(S) \subset R_2^{\times}$, so $\exists\ \varphi(t)^{-1}$)

if $\frac{a_1}{t_1} = \frac{a_2}{t_2}$, then $\exists\, t_3 \in S$: $a_1 t_2 t_3 = a_2 t_1 t_3$

$$\psi\left(\frac{a_1}{t_1}\right) = \varphi(a_1)\varphi(t_1)^{-1} = \varphi(a_1)\varphi(t_2t_3)\varphi(t_2t_3)^{-1}\varphi(t_1)^{-1}$$
$$= \varphi(a_1t_2t_3)\cdot\varphi(t_1t_2t_3)^{-1} = \varphi(a_2t_1t_3)\varphi(t_1t_2t_3)^{-1} =$$
$$= \varphi(a_2)\varphi(t_1t_3)\cdot\varphi(t_1t_3)^{-1}\varphi(t_2)^{-1} = \varphi(a_2)\varphi(t_2)^{-1}$$
$$= \psi\left(\frac{a_2}{t_2}\right) \Rightarrow \psi \text{ is well-defined}$$

$\psi\circ\lambda(a) = \psi\left(\frac{a}{1}\right) = \varphi(a)$ – commutativity of the diagram

Uniqueness of ψ – exercise $\square$

<u>Examples</u> 1) Assume that $S \subset R$ is mult. closed and $0 \in S$. Then $\forall \frac{a}{t} \in S^{-1}R$

$\frac{a}{t} = 0 = \frac{0}{1}$, because $0\cdot a = 0$
(here we take $t_3 = 0$)

This means that $S^{-1}R = 0$ (i.e. $1 = 0$ in this ring)

Conversely, if $1=0$ in $S^{-1}R$, then this means $\exists t_3 \in S$:

$$1 \cdot t_3 = 0 \cdot t_3 \Rightarrow t_3 = 0$$

$\Rightarrow 0 \in S$.

So: $S^{-1}R = 0 \iff 0 \in S$.

2) Assume that R is an int. domain and $S = R \setminus \{0\}$. Then S is mult. closed. Then $S^{-1}R = \{\frac{a}{t} \mid t \neq 0\}$ is the field of fractions of R.

3) Assume that $a \in R$ is not nilpotent, i.e. $a^n \neq 0$ $\forall n \geq 0$. Consider

$$S = \{a^n \mid n \geq 0\}$$

Then $S^{-1}R = \{\frac{x}{a^n} \mid x \in R,\ n \geq 0\}$

$S^{-1}R$ is also denoted $R[1/a]$

4) Assume that $p \subset R$ is a prime ideal. Let $S = R \setminus p$. Then $1 \in S$,

$a, b \in S \Rightarrow a \notin p,\ b \notin p \Rightarrow ab \notin p$ (because p is prime) $\Rightarrow ab \in S$

Also $0 \notin S$.

The ring $S^{-1}R$ is denoted $R_{\mathfrak{p}}$ and is called localization of R at $\mathfrak{p}$.

5) $R = \mathbb{Z}$. Take $p \in \mathbb{Z}$ a prime number

$$\mathbb{Z}[1/p] = \{ a/p^n \mid a \in \mathbb{Z},\ n \geq 0 \}$$

$$\mathbb{Z}_{(p)} = \{ a/b \mid a, b \in \mathbb{Z} \text{ and } b \text{ not divisible by } p \}$$

6) $R = K \times K$ where K is a field.

$$\mathfrak{p} = \{ (a,0) \mid a \in K \} \qquad R/\mathfrak{p} \simeq K$$

$$\Rightarrow \mathfrak{p} \text{ is prime}$$

$$S = R \setminus \mathfrak{p} = \{ (a,b) \in R \mid b \neq 0 \}$$

$$S^{-1}R = \left\{ \frac{(a,b)}{(c,d)} \mid d \neq 0 \right\}$$

$$\frac{(a,b)}{(c,d)} = \frac{(a,b)(0,d^{-1})}{(c,d)(0,d^{-1})} = \frac{(0,bd^{-1})}{(0,1)}$$

$\forall$ element from $S^{-1}R$ can be written as $\dfrac{(0,x)}{(0,1)}$; $\qquad \dfrac{(0,x_1)}{(0,1)} = \dfrac{(0,x_2)}{(0,1)}$

$$\Leftrightarrow \exists (c,d) \in S :$$

$$(0,x_1)\cdot(c,d) = (0,x_2)\cdot(c,d) \Leftrightarrow x_1 d = x_2 d$$

$$\underset{d \neq 0}{\Leftrightarrow} x_1 = x_2$$

So the representation $\frac{(0,x)}{(0,1)}$ is unique.

$$S^{-1}R \xrightarrow{\sim} K$$
$$\frac{(0,x)}{(0,1)} \longleftrightarrow x$$

Ideals and localization

$I \subset R$ ideal, $S \subset R$ mult. closed subset

$$\lambda : R \longrightarrow S^{-1}R$$

The extension of I (i.e. the ideal $\lambda(I)\cdot S^{-1}R$) is denoted $S^{-1}I$

$$S^{-1}I = \left\{ \frac{x}{t} \,/\, x \in I,\ t \in S \right\} \subset S^{-1}R$$

Consider $\overline{S}$ = the image of S in R/I.

$\overline{S}$ is mult. closed subset $\Rightarrow$ we can localize R/I. Consider the induced

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d] & S^{-1}R \arrow[d, dashed, "\varphi"] \\ R/I \arrow[r, "\overline{\lambda}"'] & \overline{S}^{-1}(R/I) \end{tikzcd}$$

map $R \longrightarrow \overline{S}^{-1}(R/I)$

The elements $t \in S$ become invertible under this map

$\Rightarrow$ By Lemma 13.1 $\exists!$ map φ making the square commutative

Lemma 13.2 In the above setting $\ker(\varphi) \simeq S^{-1}I$, and φ is surjective, so $\bar{S}^{-1}(R/I) \simeq S^{-1}R/_{S^{-1}I}$

Proof Note: if $S \cap I \neq \emptyset$, then $0 \in \bar{S}$, $S^{-1}I = S^{-1}R$, so $\bar{S}^{-1}(R/I) = 0$ and $S^{-1}R/_{S^{-1}I} = 0$.

We may assume that $S \cap I = \emptyset$.

Denote by $\bar{x}$ the image of $x \in R$ in R/I. Then $\varphi\left(\frac{x}{t}\right) = \frac{\bar{x}}{\bar{t}}$;

$\frac{x}{t} \in \ker(\varphi) \Leftrightarrow \frac{\bar{x}}{\bar{t}} = 0$ is $\bar{S}^{-1}(R/I)$

$\Leftrightarrow \exists\ \bar{t}_1 \in \bar{S} : \bar{x} \cdot \bar{t}_1 = 0 \Leftrightarrow x \cdot t_1 \in I$
$t_1 \in S$

then $\frac{x}{t} = \frac{x t_1}{t t_1} \in S^{-1}I$

$\Rightarrow \ker(\varphi) \subset S^{-1}I$

the other inclusion is obvious

$\Rightarrow \ker(\varphi) = S^{-1}I$.

It remains to prove that φ is surj.

$\frac{\bar{x}}{\bar{t}} \in \bar{S}^{-1}(R/I)$ then $\frac{\bar{x}}{\bar{t}} = \varphi\left(\frac{x}{t}\right)$

here $x \in R$, $t \in S$

$\square$

<u>Lemma 13.3</u> Assume R is an int. domain and $S \subset R$ mult. closed subset, $0 \notin S$. Then $S^{-1}R$ is also an int. domain.

<u>Proof</u> $0 \notin S \Rightarrow S^{-1}R \neq 0$

If $\frac{x_1}{t_1} \cdot \frac{x_2}{t_2} = 0$, then

$\exists t_3 \in S: \quad x_1 x_2 t_3 = 0$

Since R is an int. domain, either $x_1 = 0$ or $x_2 = 0$

but then either $\frac{x_1}{t_1} = 0$ or $\frac{x_2}{t_2} = 0$ $\square$

Assume now that $p \subset R$ is prime. $p \cap S = \emptyset$. Then by Lemma 13.2

$$S^{-1}R / S^{-1}p \cong \bar{S}^{-1}(R/p) \text{ is an int.}$$

domain by Lemma 13.3. Hence $S^{-1}p$ is also a prime ideal.

Prop. 13.4 The localization $\lambda: R \to S^{-1}R$ induces a bijection

$$\operatorname{Spec}(S^{-1}R) \xleftrightarrow{1:1} \{p \in \operatorname{Spec} R \mid p \cap S = \emptyset\}$$

Proof

$$\begin{aligned} \{p \in \operatorname{Spec} R \mid p \cap S = \emptyset\} &\longrightarrow \operatorname{Spec}(S^{-1}R) \\ \rotatebox{90}{$\in$}\quad & \\ p \quad &\longmapsto S^{-1}p \end{aligned}$$

The map in the other direction:

$$\begin{aligned} \operatorname{Spec}(S^{-1}R) &\longrightarrow \{p \in \operatorname{Spec}(R) \mid p \cap S = \emptyset\} \\ \mathfrak{q} \quad &\longmapsto \lambda^{-1}(\mathfrak{q}) \end{aligned}$$

Note: $\lambda^{-1}(\mathfrak{q}) \cap S = \emptyset$, because if $x \in S$, $\lambda(x) \in \mathfrak{q}$, then $\lambda(x)$ is invertible and $\mathfrak{q} = (1)$ – contradiction

The maps are mutually inverse:

$$S^{-1}(\lambda^{-1}(\mathfrak{q})) = \left\{\frac{x}{t} \;\middle|\; x \in \lambda^{-1}(\mathfrak{q}), t \in S\right\} = \mathfrak{q}$$

$$\lambda^{-1}(S^{-1}p) = \left\{x \in R \;\middle|\; \exists \frac{x_1}{t_1} \in S^{-1}p, \text{ s.t. } \frac{x}{1} = \frac{x_1}{t_1}\right\} = p$$

$$\frac{x}{1} = \frac{x_1}{t_1} \iff \exists t_2 \in S: \; x t_1 t_2 = x_1 t_2 \in p$$

$$\Rightarrow x \in p \qquad \square$$

<u>Cor.</u> 1) $\operatorname{Spec} R[1/x] = \{\mathfrak{p} \in \operatorname{Spec} R \mid x \notin \mathfrak{p}\}$

2) $\operatorname{Spec} R_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subset \mathfrak{p}\}$

(Here $S = R \setminus \mathfrak{p}$)

<u>Example:</u> 1) $p \in \mathbb{Z}$ prime

$\operatorname{Spec} \mathbb{Z}[1/p] = \{q \in \mathbb{Z} \text{ prime} \mid p \neq q\} \cup \{(0)\}$

2) $p \in \mathbb{Z}$ prime

$\operatorname{Spec} \mathbb{Z}_{(p)} = \{q \in \mathbb{Z} \text{ prime} \mid p = q\} \cup \{(0)\}$

$= \{(p), (0)\}$

$\mathbb{Z}_{(p)}$ has only one non-zero prime ideal.