17. Absolute values and completions (continuation)

Recall: $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is an abs. val. if

1) $|x| = 0 \Leftrightarrow x = 0$ 2) $|xy| = |x| \cdot |y|$ 3) $|x+y| \leq |x| + |y|$

if $|x+y| \leq \max\{|x|, |y|\}$ then $|\cdot|$ is non-archimedian, otherwise – archimedian

Note: $|1| = |1^2| = |1|^2 \Rightarrow |1| = 1$

$|x| = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is the triv. abs. value.

Rem: K – finite field, then the only abs. val. on K is the trivial one: $K^\times$ is cyclic, for a generator $x \in K^\times$ we have $1 = |x^n| = |x|^n$ where $n = |K^\times| \Rightarrow |x| = 1$

We will always assume char $K = 0$.

Lemma 17.1 An abs. val. $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is non-archimed. if and only if $\forall n \in \mathbb{Z}$ $|n| \leq 1$

Proof if $|\cdot|$ is non-arch $\Rightarrow |n| =$

$= |1 + (n-1)| \leq \max\{|1|, |n-1|\} \underset{\text{induction on } n.}{=} 1$

Conversely: assume $|n| \leq 1 \quad \forall n \in \mathbb{Z}$

$\forall x, y \in K$ we have

$$(x+y)^d = \sum_{k=0}^{d} \binom{k}{d} x^k y^{d-k}$$

$$|x+y|^d \leq \sum_{k=0}^{d} \underbrace{\left|\binom{k}{d}\right|}_{\leq 1} \cdot |x|^k \cdot |y|^{d-k}$$

$$\leq d \cdot \max_{k=0,\dots,d} \left(|x|^k \cdot |y|^{d-k}\right)$$

Assume $|x| \geq |y|$

Then $|x+y|^d \leq d \cdot |x|^d$

$$\Rightarrow |x+y| \leq \underbrace{d^{1/d}}_{\xrightarrow{d\to\infty} 1} |x| \xrightarrow[d\to\infty]{} |x|$$

analogously for $|y| > |x|$: $|x+y| \leq |y|$ $\square$

<u>Recall:</u> 2 abs. val. $|\cdot|_1$ and $|\cdot|_2$ are equiv. ($|\cdot|_1 \sim |\cdot|_2$) if $\exists \lambda > 0$: $\forall x \in K \quad |x|_1 = |x|_2^{\lambda}$. Equiv. class of abs. values = a place of K

Lemma 17.1 $\Rightarrow$ equivalent abs. values are either both arch. or both non-arch.

An abs. val. defines a metric on K

$d(x,y) = |x-y| \Rightarrow$ a topology on K

open subsets are unions of open balls

$$B_r(x) = \{y \in K \mid |x-y| < r\}$$

<u>Prop. 17.2</u> $|\cdot|_1 \sim |\cdot|_2 \iff |\cdot|_1$ and $|\cdot|_2$ define the same topology on K.

<u>Proof</u> $\Rightarrow$ clear: if $|\cdot|_1 \sim |\cdot|_2$, then the open balls are the same.

$\Leftarrow$ We assume that $|\cdot|_1$ and $|\cdot|_2$ define the same topology; Note:

$$|x|_1 < 1 \iff |x^n|_1 \xrightarrow[n\to\infty]{} 0$$

$$\iff x^n \xrightarrow[n\to\infty]{} 0 \iff |x^n|_2 \xrightarrow[n\to\infty]{} 0$$

convergence in the topology defined by $|\cdot|_1$ or $|\cdot|_2$

$$\Updownarrow$$

$$|x|_2 < 1$$

If $|\cdot|_1$ is trivial, then $\forall\, 0 \neq x \in K$

$|x|_1 = 1 \Rightarrow |x|_2 = 1$ (if $|x|_2 < 1$ then $|x|_1 < 1$; if $|x|_2 > 1$, then $|1/x|_2 < 1 \Rightarrow |1/x|_1 < 1$)

$\Rightarrow |\cdot|_2$ is trivial

Assume $|\cdot|_1$ is non-trivial. Then $\exists x_0 \in K$ $|x_0|_1 > 1$. Then $|x_0|_2 > 1$,

let $|x_0|_1 = |x_0|_2^{\lambda}$, i.e. $\lambda = \dfrac{\log|x_0|_1}{\log|x_0|_2} > 0$

Take $0 \neq x \in K$, then

$$|x|_1 = |x_0|_1^{\alpha} \text{ for some } \alpha \in \mathbb{R}$$

If $\frac{m}{n} < \alpha$ for some $m, n \in \mathbb{Z}$

then $|x|_1 = |x_0|_1^{\alpha} > |x_0|_1^{\frac{m}{n}}$

$$\Rightarrow |x^n|_1 > |x_0^m|_1 \Rightarrow \left|\frac{x_0^m}{x^n}\right|_1 < 1$$

$$\Rightarrow \left|\frac{x_0^m}{x^n}\right|_2 < 1 \Rightarrow |x|_2 > |x_0|_2^{m/n}$$

Analogously, if $\alpha < \frac{m}{n} \Rightarrow |x|_2 < |x_0|_2^{m/n}$

$$\Rightarrow |x|_2 = |x_0|_2^{\alpha}$$

Then $|x|_1 = |x_0|_1^{\alpha} = |x_0|_2^{\lambda\alpha} = |x|_2^{\lambda}$

$$\Rightarrow |\cdot|_1 \sim |\cdot|_2 \qquad \square$$

We denote by $|\cdot|_\infty$ the usual abs. val on $\mathbb{Q}$.

Prop. 17.3 Let $\|\cdot\|$ be an archimed. abs. val. on $\mathbb{Q}$. Then $\|\cdot\| \sim |\cdot|_\infty$

Proof Enough to prove: $\exists \lambda > 0: \forall n \in \mathbb{Z}$
$\|n\| = |n|^\lambda$

Step 1 We claim that $\forall n \in \mathbb{Z}, n > 1$
$\|n\| > 1$. Assume not, let
$\|n\| \leq 1$ for some n.

For $m \in \mathbb{Z}$, $m \geq 0$ we can write

$$(*) \quad m = \sum_{k=0}^{N} a_k n^k, \quad 0 \leq a_k < n, \text{ where}$$

N is the smallest integer s.t. $m < n^{N+1}$

i.e. $N = \left\lfloor \frac{\log m}{\log n} \right\rfloor \leq \frac{\log m}{\log n}$

Let $A = \max\limits_{i=1,\dots,n-1} \|i\|$. Then

$$\|m\| \leq A(1+N) \leq A\left(1 + \frac{\log m}{\log n}\right)$$

Then $\|m^d\| = \|m\|^d \leq A\left(1 + d\frac{\log m}{\log n}\right)$

$$\Rightarrow \|m\| \leq A^{1/d}\left(1 + d\frac{\log m}{\log n}\right)^{1/d} \xrightarrow[d\to\infty]{} 1$$

$\Rightarrow \forall m \in \mathbb{Z} \quad \|m\| \leq 1 \overset{\text{Lemma 17.1}}{\Longrightarrow} \|\cdot\|$ is non-arch.

contradiction.

<u>Step 2</u> Take arbitrary $n>1$, $m>1$

Use $(*)$:

$$\|m\| \le A(1+N)\|n\|^{\frac{\log m}{\log n}} \le A\left(1+\frac{\log m}{\log n}\right)\|n\|^{\frac{\log m}{\log n}}$$

and $$\|m^d\| = \|m\|^d \le A\left(1+d\frac{\log m}{\log n}\right)\|n\|^{d\frac{\log m}{\log n}}$$

$$\Rightarrow \|m\| \le A^{1/d}\left(1+d\frac{\log m}{\log n}\right)^{1/d}\cdot\|n\|^{\frac{\log m}{\log n}} \xrightarrow[d\to\infty]{} \|n\|^{\frac{\log m}{\log n}}$$

$$\Rightarrow \|m\|^{\frac{1}{\log m}} \le \|n\|^{\frac{1}{\log n}}$$

exchanging m and n we get reverse inequal. $\Rightarrow$ $\|m\|^{\frac{1}{\log m}} = \|n\|^{\frac{1}{\log n}}$

$\forall m,n \geq 2$

Denote this constant by C

Then $$\|m\| = C^{\log m} = e^{\log m\cdot\log C} = m^{\log C}$$

let $\lambda = \log C > 0$

$$\Rightarrow \forall m \geq 2 \quad \|m\| = m^\lambda$$

Use $\|-1\| = 1 \Rightarrow \forall m\in\mathbb{Z}\quad \|m\| = |m|^\lambda$

$\square$

Recall: if $|\cdot|_v$ is an abs. val. on K there exists the completion K_v: it is the unique (up to isomorph.) field, s.t. $K \hookrightarrow K_v$ and $|\cdot|_v$ extends to K_v making it complete metric space with K a dense subset. E.g. $\mathbb{Q}_\infty = \mathbb{R}$

Assume K is a number field, and $|\cdot|_v$ is an archimedian abs. val.

$|\cdot|_v|_{\mathbb{Q}} \sim |\cdot|_\infty$ by the previous proposition

We may assume (replacing $|\cdot|_v$ by $|\cdot|_v^{\lambda}$) that $|\cdot|_v|_{\mathbb{Q}} = |\cdot|_\infty$

K_v contains $\mathbb{R}$ = the closure of $\mathbb{Q}$ in K_v.

We get: $K \subset K_v$, $\mathbb{R} \subset K_v$

Let $K' = K \cdot \mathbb{R}$ (the subfield of K_v generated by K, $\mathbb{R}$)

K is of fin degree $/\mathbb{Q} \Rightarrow /\mathbb{R}$

$\Rightarrow K'$ is a fin. ext. of $\mathbb{R}$

$\Rightarrow K' = \mathbb{R}$ or $K' = \mathbb{C}$

$K \subset K'$ and $|\cdot|_v$ is the restriction of an abs. val. on K'

<u>Lemma 17.4</u> The only abs. val. on $\mathbb{C}$ that extends the usual abs. val on $\mathbb{R}$ is the usual one $\left(z \longmapsto |z| = (z \cdot \bar{z})^{\frac{1}{2}}\right)$

<u>Proof</u> Assume we have an abs. val $|\cdot|_1$ on $\mathbb{C} / \mathbb{R}$, then $|\cdot|_1$ defines a norm on $\mathbb{C}$ as a 2-dim v.sp. $/\mathbb{R}$ all norms of fin dim. $\mathbb{R}$- v. spaces are equivalent $\Rightarrow$ define the same topology $\Rightarrow$ by Prop 17.2 $|z|_1 = |z|^\lambda$ $\forall z \in \mathbb{C}$ and some $\lambda > 0$,
$\lambda = 1$ because $|x|_1 = |x|$ for $x \in \mathbb{R}$ $\square$

<u>Conclusion:</u> $K \hookrightarrow K' = \mathbb{R}$ or $\mathbb{C}$ with the standard abs. value, and $|\cdot|_v$ is the restriction.

$\Rightarrow$ All archimedian abs. val. on K are of the form

$|x|_\sigma = |\sigma(x)|$ for some field embedding $\sigma : K \hookrightarrow \mathbb{C}$

They are equiv. for complex-conj. embeddings

These abs. values define the "infinite places" of K

<u>Rem.</u> More generally, all archimedian abs. values on arbitrary fields are obtained the same way: $K \hookrightarrow \mathbb{C}$ and the abs. val restricts from $\mathbb{C}$

In part the only fields complete w.r.t. an archimed. abs. val are $\mathbb{R}$ and $\mathbb{C}$.