8. Noetherian rings

Recall from last time: a ring R is Noetherian if $\forall$ ideal $I \subset R$ is finitely generated, i.e. $\exists x_1, \dots, x_n \in I$, s.t. $\forall x \in I$ $\quad x = \sum_{i=1}^{n} a_i x_i$ for some $a_i \in R$

Examples 1) $\mathbb{Z}$, $K[X]$ – PID, in particular they are Noetherian

2) $\mathcal{O}_K$ – integers in a number field K (see lecture 6)

3) Assume R – Noetherian, $I \subset R$ an ideal

Then R/I also Noetherian: let

$\varphi: R \twoheadrightarrow R/I$; $\quad J \subset R/I$ an ideal

$\Rightarrow \varphi^{-1}(J) \subset R$ ideal in $R \Rightarrow \varphi^{-1}(J)$ is fin. gen. by $x_1 \dots x_n \in \varphi^{-1}(J)$

Then $J = (\varphi(x_1), \dots, \varphi(x_n))$

4) A non-example: $R = K[X_1, X_2 \dots]$ a polynom. ring in ∞ number of variables.

$I = (X_1, X_2, \dots)$ not fin. gen. $\Rightarrow R$ is <u>not</u> Noeth.

Thm 8.1 (Hilbert's basis theorem) If R is Noetherian, then $R[x]$ is also Noeth.

Proof Let $I \subset R[x]$ an ideal.

Step 1 for $f \in R[x]$, $f = a_0 x^n + a_1 x^{n-1} + \dots + a_n$, $a_i \in R$ let $\ell(f) = \begin{cases} a_0, & \text{if } \deg(f) = n \\ 0 & \text{if } f = 0 \end{cases}$

– the "leading coefficient"

Define subsets in R:

$$J_k = \{ \ell(f) \mid f \in I \text{ and } \deg(f) \leq k \}, \quad k = 0, 1, \dots$$

$$J_\infty = \{ \ell(f) \mid f \in I \}$$

Then J_k and J_∞ are ideals:

$0 = \ell(0) \in J_k$; if $b_1 = \ell(f_1)$, $b_2 = \ell(f_2)$

$b_1, b_2 \in J_k$ $\quad f_1 = b_1 x^{d_1} + \dots \quad d_1 \leq k$

$\quad f_2 = b_2 x^{d_2} + \dots \quad d_2 \leq k$

$-b_1 = \ell(-f_1) \in J_k$

We may assume $d_1 \leq d_2$

$$b_1 + b_2 = \begin{cases} 0 \in J_k, & \text{or} \\ \ell(\underbrace{x^{d_2 - d_1} f_1 + f_2}_{(b_1 + b_2) x^{d_2} + \dots}) \in J_k \end{cases}$$

$a \in R \qquad a \cdot b_1 = \ell(a \cdot f_1) \in J_k$

$\Rightarrow J_k$ is an ideal;

analogously for J_∞

R is Noetherian $\Rightarrow J_k$ and J_∞ are fin. generated:

$$J_k = (\ell(f_{k,1}), \dots, \ell(f_{k,m_k})), \; f_{k,j} \in I$$

$$J_\infty = (\ell(f_{\infty,1}), \dots, \ell(f_{\infty,n})), \; f_{\infty,i} \in I$$

<u>Step 2</u> Define $d = \max\limits_{i=1\dots n}(\deg(f_{\infty,i}))$

and

$$I' = (f_{\infty,i} \,;\, f_{k,j})_{\substack{i=1\dots n \\ k=0,\dots,d-1 \\ j=1\dots m_k}}$$

I' is finitely gen. ideal in $R[x]$; $\quad I' \subset I$

Claim: $I' = I$. Let $g \in I$

We use induction on $\deg(g)$

Base of ind: $\deg(g) = 0$; then $g = \ell(g) \in J_0$

$\Rightarrow g$ is in the ideal gen. by $f_{0,1}, \dots, f_{0,m_0}$

$\Rightarrow g \in I'$

Step of induction: assume $\deg(g) = k$ and $\forall h \in I$ with $\deg(h) < k$ $h \in I'$

Case 1: $k \leq d-1$

$\ell(g) \in J_k \implies$ we have:

$$\ell(g) = \sum_{j=1}^{m_k} a_j \, \ell(f_{k,j}), \quad a_j \in R$$

Consider

$$h = g - \sum_j a_j \underbrace{X^{k-\deg f_{k,j}} \cdot f_{k,j}}_{\in I'}$$

$\deg(h) < k \implies$ by inductive assumption

$h \in I' \implies g \in I'$

Case 2: $k \geq d$ $\quad \ell(g) \in J_\infty$

again, $\ell(g) = \sum_i a_i \, \ell(f_{\infty,i})$

Note: $\deg(f_{\infty,i}) \leq k$

Consider
$$h = g - \sum_i a_i X^{k-\deg(f_{\infty,i})} \cdot f_{\infty,i}$$

then $\deg(h) < k \implies h \in I'$ (by inductive assumption)

and then $g \in I'$

$\implies I' = I$, so I is fin. gen. $\square$

Def R is called a finitely gen. K-algebra if $\exists$ a surjective ring homomorph. $K[X_1, \dots X_n] \twoheadrightarrow R$ for some n.

Corollary from 8.1 Any fin. generated K-alg. is Noetherian

Proof Use Thm. 8.1 inductively:
K is Noetherian, because only $(0), (1)$ are ideals in K.
$\Rightarrow K[X_1, \dots X_n]$ is Noetherian; then
$R = K[X_1, \dots X_n] / I$ for some ideal I
$\Rightarrow R$ is also Noeth. (see example 3) $\square$

Let M be a module over some ring R

Def M is fin. generated if $\exists$ a surj. morphism of R-modules

$$R^{\oplus n} \twoheadrightarrow M$$

for some n.

Prop 8.2 Let R be a Noeth. ring, M - fin. gen. R-module. Then any submodule $N \subset M$ is also fin. gen.

Proof $\exists\, \varphi: R^{\oplus n} \twoheadrightarrow M$ surj.
take $\varphi^{-1}(N)$; if $\varphi^{-1}(N)$ is fin. gen. by $n_1, \dots, n_k \in \varphi^{-1}(N)$, then N is generated by $\varphi(n_1), \dots \varphi(n_k)$
$\Rightarrow$ enough to assume that $M \simeq R^{\oplus n}$
We have $N \subset R^{\oplus n}$. Induction on n
Base: $n = 1$. $N \subset R$ a submodule
$\Rightarrow N$ is an ideal $\overset{\text{Thm. 8.1}}{\Longrightarrow} N$ is fin. generated.

Induction step: $\alpha: R \hookrightarrow R^{\oplus n}$
$$x \longmapsto (x, \underbrace{0 \dots 0}_{n-1})$$

Let $N' = N \cap \alpha(R)$.
$N'' = N / N'$
$$N' = \ker\left(R^{\oplus n} \twoheadrightarrow R^{\oplus n}/\alpha(R)\right) \cap N$$

so $N'' \subset R^{\oplus n}/\alpha(R) \simeq R^{\oplus n-1}$

$N' \subset R$ fin. generated by $n_1', \dots, n_k' \in N'$

$N'' \subset R^{\oplus n-1}$ also fin. gen. by inductive assumption

N'' gen. by $n_1'', \dots, n_\ell'' \in N''$

let $\bar{n}_i''$ be some preimages of n_i'' under the map $\psi: N \twoheadrightarrow N''$

Claim: N is generated by

$n_1', \dots, n_k', \bar{n}_1'', \dots, \bar{n}_\ell''$

Let $x \in N \Rightarrow \psi(x) = \sum_i a_i n_i''$, $a_i \in R$

consider $x' = x - \sum a_i \bar{n}_i''$

then $x' \in \ker \psi = N'$

$\Rightarrow x' = \sum_j b_j n_j'$

$\Rightarrow x = \sum_i a_i \bar{n}_i'' + \sum_j b_j n_j'$ $\square$

<u>Def</u> Let R be a ring, M - R-module. M satisfies <u>ascending chain condition</u> (ACC) if any sequence of submodules

$$M_1 \subset M_2 \subset M_3 \subset \dots \subset M$$

stabilizes, i.e. $\exists n \geq 1$ s.t.

$$\forall i \geq 0 \quad M_{n+i} = M_n$$

<u>Prop 8.3.</u> Let M be an R-module

TFAE:

1) $\forall$ submodule $N \subset M$ is fin. gen.

2) M satisfies ACC.

<u>Proof</u> 1) $\Rightarrow$ 2)

let $M_1 \subset M_2 \subset \dots$ be a chain of submodules

define $N = \bigcup_{i=1}^{\infty} M_i$; $N = (m_1, \dots, m_k)$

for some $m_i \in N$. Then $\exists n$, s.t.

$\forall i = 1 \dots k \quad m_i \in M_n \Rightarrow N = M_n$

$\Rightarrow M_{n+i} = M_n \quad \forall i \geq 0.$

2) $\Rightarrow$ 1) Let $N \subset M$ a submodule.

Assume N is not fin. gen. Construct

a chain of submodules inductively

$N_1 = (x_1)$ for some $0 \neq x_1 \in N$

such x_1 exist by assumption

induction step: we have $N_1 \subsetneq \dots \subsetneq N_{k-1}$,

$N_i = (x_1, \dots, x_i)$

by assumption $N \neq N_{k-1}$, so

$\exists\, x_k$, s.t. $x_k \in N \setminus N_{k-1}$

then define $N_k = (x_1, \dots, x_k)$

$\Rightarrow$ get $N_1 \subsetneq N_2 \subsetneq \dots$ infinite chain which does not stabilize $\Rightarrow$ contradicts ACC for M $\square$

<u>Cor 1</u> Assume R is Noetherian, and $I_1 \subset I_2 \subset \dots$ a chain of ideals

Then $\exists\ n \geq 1$ s.t. $I_{n+i} = I_n\ \forall i \geq 0$

<u>Proof</u>: clear $\square$

<u>Cor 2</u> Assume R is Noetherian, and $\mathcal{C}$ - any non-empty set of ideals in R. Then $\mathcal{C}$ contains a maximal element, i.e. $\exists\ I \in \mathcal{C}$, s.t. $\forall J \in \mathcal{C}$

$I \subset J \Rightarrow I = J$

Proof Assume the assertion is false, so $\nexists$ maximal element in $\mathcal{C}$

Construct a chain of ideals inductively

$\exists \quad I_1 \in \mathcal{C}$ because $\mathcal{C} \neq \emptyset$

I_1 is not a maximal element of $\mathcal{C}$

$\Rightarrow \exists\, I_2 \in \mathcal{C}$, s.t. $I_1 \subsetneq I_2$

I_2 is also not maximal $\Rightarrow$ we can go on $\Rightarrow$ get $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$

does not stabilize $\Rightarrow$ contradicts corollary 1 $\square$