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Algebraic Number Theory

Exercise sheet 1

Solutions should be submitted online before 27.04.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 1.1. (3 points) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain. Is this true for $\mathbb{Z}[\sqrt{-3}]$?

Exercise 1.2. (3 points) For arbitrary $n \in \mathbb{Z}$, determine the conditions under which n can be represented as the sum of two squares.

Exercise 1.3. (3 points) Find all integral solutions of the equation $X^2 + Y^2 = Z^2$.

Exercise 1.4. (3 points) Let K be a field and $A \subset K^\times$ a finite subgroup. Prove that A is cyclic. Deduce that $\mathbb{F}_q^\times$ is cyclic, where $\mathbb{F}_q$ is a finite field.

Exercise 1.5. (2 + 2 points)

1. Let $\mu_n \subset \mathbb{C}^\times$ be the group of n-th roots of unity, i.e. the group of all $x \in \mathbb{C}^\times$ such that $x^n = 1$. The generators of μ_n are called *primitive* n-th roots of unity. How many primitive n-th roots of unity are there?

2. Let $\omega \in \mu_n$ be a primitive n-th root of unity. Prove that $\mathbb{Q}(\omega)$ is a Galois extension of $\mathbb{Q}$ and that $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ is abelian.