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Algebraic Number Theory

Exercise sheet 10

Solutions should be submitted online before 29.06.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 10.1. (2 points) Let $K \subset L$ be a Galois extension of number fields. Assume that the order of $\mathrm{Gal}(L/K)$ is prime. Prove that any prime ideal $\mathfrak{p} \subset \mathfrak{o}_K$ that is ramified in L is totally ramified in L.

Exercise 10.2. (4 points) Let $K \subset L$ be a Galois extension of number fields, $\mathfrak{p} \subset \mathfrak{o}_L$ a prime ideal. Let G be the Galois group of L over K, $G_\mathfrak{p}$ the decomposition group of $\mathfrak{p}$ and $I_\mathfrak{p}$ the inertia group of $\mathfrak{p}$. The decomposition field $L_\mathfrak{p}$ is the subfield of L fixed by $G_\mathfrak{p}$. Consider the inertial field $F_\mathfrak{p}$, i.e. the subfield of L fixed by $I_\mathfrak{p}$. Prove that $\mathfrak{p} \cap \mathfrak{o}_{L_\mathfrak{p}}$ is unramified in the extension $L_\mathfrak{p} \subset F_\mathfrak{p}$, and $\mathfrak{p} \cap \mathfrak{o}_{F_\mathfrak{p}}$ is totally ramified in the extension $F_\mathfrak{p} \subset L$.

Exercise 10.3. (2 points) Determine which primes divide the discriminant of the quadratic field $\mathbb{Q}(\sqrt{d})$ and hence ramify in it. Compare this to the results you obtained in Exercise 8.3 from the exercise sheet 8.

Exercise 10.4. (2 + 2 points) Let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive p-th root of unity for some prime number p.

1. Prove that $|N_{K/\mathbb{Q}}(\zeta - 1)| = p$
2. Compute $Tr_{K/\mathbb{Q}}(\zeta^i)$ and $Tr_{K/\mathbb{Q}}(\zeta^i - 1)$ for all i.