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Algebraic Number Theory

Exercise sheet 12

Below is a list of questions and exercises that should serve as a reminder and a preparation for the exam. You should not submit solutions for this exercise sheet. The questions are supposed to be simple and you should have no problem answering them quickly. If you find some questions problematic, it possibly indicates that you should review the relevant parts of the lectures.

1. List all units of the ring $\mathbb{Z}[i]$.
2. Decompose the ideal $(2) \subset \mathbb{Z}[i]$ into a product of prime ideals.
3. Determine the order of the automorphism group of the field $\mathbb{Q}[X]/(X^3 - 2)$ over $\mathbb{Q}$.
4. List all subfields of the field $\mathbb{Q}(\mu_5)$, where μ_5 is a primitive 5-th root of unity.
5. Compute the trace and the norm of $3 - \sqrt{-5} \in \mathbb{Q}(\sqrt{-5})$.
6. Is it true that any number field K is of the form $K = \mathbb{Q}(\alpha)$, where α is an algebraic integer?
7. Assume that K is a number field. What is the integral closure of $\mathbb{Q}$ in K?
8. Let K be the splitting field of $X^3 - 2 \in \mathbb{Q}[X]$. What is the rank of $\mathfrak{o}_K$ as a $\mathbb{Z}$-module?
9. Does there exist a number field K with $d_K = 23$?
10. Does there exist a number field K with $|d_K| = 23^n$ for some $n > 1$?
11. List all prime ideals of the ring $\mathbb{Z}[i]/(2)$.
12. Is it true that any finitely generated $\mathbb{Z}$-submodule of $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$?
13. Find an example of a Dedekind domain that has exactly two non-zero prime ideals.
14. Is the group of fractional ideals of a number field finitely generated?
15. Give an example of a number field with a non-trivial ideal class group.
16. Is it true that for any number field K there are only finitely many fractional ideals whose norm is bounded from above?
17. Let $K = \mathbb{Q}(\mu)$ with μ a primitive n-th root of unity, $n \geqslant 2$. Find the numbers r_1 and r_2 of real embeddings $\sigma_i \colon K \hookrightarrow \mathbb{R}$, respectively pairs of complex-conjugate embeddings $\sigma_j, \overline{\sigma}_j \colon K \hookrightarrow \mathbb{C}$.
18. Describe the group of units of $\mathfrak{o}_K$ for $K = \mathbb{Q}(\mu)$ with μ a primitive n-th root of unity.
19. Describe the group of units of a discrete valuation ring R.
20. Let K be Galois over $\mathbb{Q}$ with $[K : \mathbb{Q}] = 15$. Can one find a prime ideal $\mathfrak{p} \subset \mathfrak{o}_K$ with $\mathfrak{o}_K/\mathfrak{p} \simeq \mathbb{F}_4$?
21. Assume that a prime p does not ramify in a Galois extension $K/\mathbb{Q}$ and $\mathfrak{p} \subset \mathfrak{o}_K$ lies over p. Is it true that the decomposition group $\mathrm{Gal}(K/\mathbb{Q})_{\mathfrak{p}}$ is cyclic?
22. Let $K_1 = \mathbb{Q}(\sqrt{5})$, $K_2 = \mathbb{Q}(\sqrt{7})$. Describe the ring $K_1 \otimes_{\mathbb{Q}} K_2$.
23. Let $K = \mathbb{Q}(\sqrt{-5})$. Describe the ring $K \otimes_{\mathbb{Q}} K$.