Dr. Andrey Soldatenkov
Jan Hesmert

Humboldt Universität zu Berlin
Summer term 2020

Algebraic Number Theory

Exercise sheet 3

Solutions should be submitted online before 11.05.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 3.1. (2 + 2 points) Compute discriminants of the following fields (you may use results of Exercise 2.3 from the previous exercise sheet):

1. $\mathbb{Q}(\sqrt{7})$
2. $\mathbb{Q}(\sqrt{-11})$

Exercise 3.2. (3 + 2 points) Let $R \subset S$ be an integral ring extension.

1. Assume that the ring S is an integral domain. Prove the R is a field if and only is S is a field.
2. Let $\mathfrak{p} \subset S$ be a prime ideal. Deduce from the first part of the exercise that $\mathfrak{p}$ is maximal if and only if $R \cap \mathfrak{p}$ is maximal.

Exercise 3.3. (4 + 3 + 2 points) Consider the ring $\mathbb{Z}[\sqrt{-5}]$ and two ideals $\mathfrak{p} = (2, 1 + \sqrt{-5})$ and $\mathfrak{q} = (3, 1 + \sqrt{-5})$ in it. Denote by $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ the ideals obtained by taking complex conjugates of the elements of $\mathfrak{p}$ and $\mathfrak{q}$.

1. Prove that $\mathfrak{p}$ and $\mathfrak{q}$ are prime but not principal. Prove that $\mathfrak{p} = \overline{\mathfrak{p}}$.
2. Prove that $\mathfrak{p}\mathfrak{q} = (1 + \sqrt{-5})$ and $\mathfrak{p}\overline{\mathfrak{q}} = (1 - \sqrt{-5})$
3. Prove that $(6) = \mathfrak{p}^2\mathfrak{q}\overline{\mathfrak{q}}$