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Algebraic Number Theory

Exercise sheet 4

Solutions should be submitted online before 18.05.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 4.1. (3 points) Let $K = \mathbb{Q}(\omega)$, where ω is a primitive 3-rd root of unity. Find the minimal polynomial of ω, the discriminant of K and describe the ring of integers $\mathfrak{o}_K$.

Exercise 4.2. (3 + 3 points) Let R be a principal ideal domain with field of fractions K.

1. Let $b \in K$ be integral over R. Prove that b can be expressed as $b = a_1/a_2$ with $a_1, a_2 \in R$ and $(a_1, a_2) = R$. Using the expression of integral dependence for b, show that $(a_2) = R$. Deduce that R is integrally closed in K.
2. Prove that any non-zero prime ideal in R is maximal. Deduce that R is a Dedekind domain.

Exercise 4.3. (5 points). Which of the following rings are Dedekind domains? Explain your answer in each case.

1. $\mathbb{Z} \times \mathbb{Z}$;
2. $\mathbb{Z}[X]/(X^2 + 3)$;
3. $\mathbb{F}_{11}[X]$;
4. $\mathbb{R}[X, Y]$;
5. $\mathbb{C}[X, Y]/(X^5 + Y - 13)$