Dr. Andrey Soldatenkov
Jan Hesmert

Humboldt Universität zu Berlin
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Algebraic Number Theory

Exercise sheet 5

Solutions should be submitted online before 25.05.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 5.1. (2 + 2 + 2 points) Let R be a Dedekind domain with field of fractions K.

1. Let $\mathfrak{p}_1, \mathfrak{p}_2 \subset R$ be non-zero prime ideals, $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Prove that $\mathfrak{p}_1 + \mathfrak{p}_2 = R$.
2. Under the same assumptions as above, prove that $\mathfrak{p}_1^{n_1} + \mathfrak{p}_2^{n_2} = R$ for any $n_1, n_2 \geqslant 1$. Hint: using part 1, find two elements $a \in \mathfrak{p}_1$, $b \in \mathfrak{p}_2$, such that $a + b = 1$; take sufficiently big power of the last equality.
3. Let $\mathfrak{a} \subset R$ be an arbitrary ideal. Prove that
$$R/\mathfrak{a} \simeq \prod_i R/\mathfrak{p}_i^{n_i}$$
for some distinct prime ideals $\mathfrak{p}_i$, integers $n_i \geqslant 1$, and this decomposition is unique up to reordering of the factors. Hint: apply Chinese Remainder Theorem.

Exercise 5.2. (3 + 3 + 3 points) Let $K = \mathbb{Q}(\sqrt{-6})$ and $\mathfrak{p}_1 = (2, \sqrt{-6})$, $\mathfrak{p}_2 = (3, \sqrt{-6})$ two ideals in $\mathfrak{o}_K = \mathbb{Z}[\sqrt{-6}]$.

1. Prove that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime but not principal. Hint: to prove that they are prime, consider the quotient rings; to prove that they are not principal, assume the contrary and consider the norms of the elements.
2. Computing $\mathfrak{p}_1^2$ and $\mathfrak{p}_2^2$ deduce that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have order 2 as elements of $\mathrm{Cl}(K)$. Hint: the product of two ideals is generated by pairwise products of the generators of these ideals.
3. Prove that $\mathfrak{p}_1\mathfrak{p}_2 = (\sqrt{-6})$. Deduce that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ define the same element of $\mathrm{Cl}(K)$. Hint: prove that $[\mathfrak{p}_1][\mathfrak{p}_2] = 1$ in $\mathrm{Cl}(K)$; apply part 2.