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Algebraic Number Theory

Exercise sheet 6

Solutions should be submitted online before 1.06.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 6.1. (3 points) Let K be a number field. Recall that the formula $\tau(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(xy)$ defines a non-degenerate symmetric bilinear form on K considered as $\mathbb{Q}$-vector space. Assume that K has r_1 real embeddings and r_2 pairs of complex-conjugate embeddings into $\mathbb{C}$ (so that $r_1 + 2r_2 = [K : \mathbb{Q}]$). Express the signature of τ in terms of r_1 and r_2. Hint: recall that $K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as an $\mathbb{R}$-algebra. Find the signature of the trace form separately for every factor.

Exercise 6.2. (2 + 2 points) Let $K = \mathbb{Q}(\sqrt{-17})$. Find the norms of the following ideals in $\mathfrak{o}_K$:

1. $\mathfrak{p} = (3, 2 + \sqrt{-17})$
2. $\mathfrak{q} = (2\sqrt{-17})$

Exercise 6.3. (3 + 3 + 3 points) Let $K = \mathbb{Q}(\sqrt{-10})$.

1. Prove that $(2) = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p} \subset \mathfrak{o}_K$ and find the generators of $\mathfrak{p}$ explicitly. Hint: consider the quotient $\mathfrak{o}_K/(2)$ and use Chinese Remainder Theorem.
2. Prove that the ideal $\mathfrak{p}$ defined above is not principal. Deduce that the order of $[\mathfrak{p}]$ in $\mathrm{Cl}(K)$ is two. Hint: assuming that $\mathfrak{p}$ is principal consider the norm of its generator and get a contradiction.
3. Prove that the ideal (3) is prime in $\mathfrak{o}_K$. Using Minkowski's bound, deduce that $\mathrm{Cl}(K)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.