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Algebraic Number Theory

Exercise sheet 7

Solutions should be submitted online before 8.06.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 7.1. (3 + 3 + 3 points) Let K be a number field. In this exercise we will prove that any ideal in $\mathfrak{o}_K$ can be generated by two elements.

1. Let $x \in \mathfrak{o}_K$ be a non-zero element, conider the decomposition $(x) = \prod \mathfrak{p}_i^{n_i}$, where $\mathfrak{p}_i$ are distinct non-zero prime ideals. Prove that $n_i = \max\{n \geqslant 0 \mid x \in \mathfrak{p}_i^n\}$ for all i. Hint: assuming that $x \in \mathfrak{p}_j^n$ with $n > n_j$ for some j, deduce that $\prod \mathfrak{p}_i^{n_i} \subset \mathfrak{p}_j^n$ and derive a contradiction.
2. Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{o}_K$ be two non-zero ideals. Prove that there exists a non-zero $x \in \mathfrak{a}$ such that the ideals $(x)/\mathfrak{a}$ and $\mathfrak{b}$ are coprime. Hint: decompose $\mathfrak{a}$ and $\mathfrak{b}$ into products of primes; use Chinese remainder theorem to construct x, such that the primes appearing in the decomposition of $\mathfrak{b}$ do not appear in the decomposition of $(x)/\mathfrak{a}$.
3. Let $\mathfrak{a} \subset \mathfrak{o}_K$ be a non-zero ideal and $b \in \mathfrak{a}$ a non-zero element. Prove that there exists $a \in \mathfrak{a}$, such that $\mathfrak{a} = (a, b)$. Hint: apply the previous part to $\mathfrak{a}$ and $\mathfrak{b} = (b)$.

Exercise 7.2. (3 + 3 + 3 + 3 points) Let K be a number field. In this exercise we will obtain an alternative description of the norm of an ideal.

1. Let $\mathfrak{p} \subset \mathfrak{o}_K$ be a non-zero prime ideal, $\mathfrak{p} \cap \mathbb{Z} = (p)$, and $x \in \mathfrak{p}$. Prove that $N_{K/\mathbb{Q}}(x)$ is divisible by p^f for some $f \geqslant 1$. Hint: decompose the ideal (x) into a product of primes; use multiplicativity of the norm.
2. In the above setting, let q be a prime number different from p. Prove that there exists $x \in \mathfrak{p}$ such that $N_{K/\mathbb{Q}}(x)$ is not divisible by q. Hint: factorize $(q) = \prod \mathfrak{q}_i$ and use CRT to find $x \in \mathfrak{p}$ such that $x \notin \mathfrak{q}_i$ for all i.
3. Use the results obtained above to deduce that the ideal $(\mathrm{Norm}(\mathfrak{p})) \subset \mathbb{Z}$ is generated by $N_{K/\mathbb{Q}}(x)$ for all $x \in \mathfrak{p}$. Hint: we have $\mathrm{Norm}(\mathfrak{p}) = p^f$ for some $f \geqslant 1$; prove that there exists $x \in \mathfrak{p}$ as in the previous part such that $N_{K/\mathbb{Q}}(x)$ is divisible by p^f, but not by p^{f+1}.
4. More generally, for any ideal $\mathfrak{a} \subset \mathfrak{o}_K$ show that $(\mathrm{Norm}(\mathfrak{a})) \subset \mathbb{Z}$ is generated by $N_{K/\mathbb{Q}}(x)$ where $x \in \mathfrak{a}$. Hint: decompose $\mathfrak{a}$ into a product of primes.