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Algebraic Number Theory

Exercise sheet 8

Solutions should be submitted online before 15.06.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 8.1. (2 + 2 points) Find the fundamental units for the following real quadratic fields:

1. $\mathbb{Q}(\sqrt{10})$
2. $\mathbb{Q}(\sqrt{13})$

Exercise 8.2. (3 + 3 points) Let $K = \mathbb{Q}(\sqrt{d})$ with $d > 0$. Consider the Pell's equation

$$a^2 - db^2 = \pm 1.$$

At the lecture we have considered the case $d \equiv 2, 3 \pmod 4$. Assume now that $d \equiv 1 \pmod 4$, so that $\mathfrak{o}_K = \mathbb{Z}[(1+\sqrt{d})/2]$.

1. Prove that solutions to Pell's equation correspond to the elements of $\mathfrak{o}_K^\times \cap \mathbb{Z}[\sqrt{d}]$.
2. Write $u = (a_1 + b_1\sqrt{d})/2$ for the fundamental unit, where $a_1, b_1 \geqslant 1$ are of the same parity. Prove that $u^3 \in \mathbb{Z}[\sqrt{d}]$. Deduce that Pell's equation always has infinitely many solutions.

Exercise 8.3. (5 points) Let $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Determine all prime numbers that ramify in $\mathfrak{o}_K$.