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Algebraic Number Theory

Exercise sheet 9

Solutions should be submitted online before 22.06.20 via the Moodle page of the course:
`https://moodle.hu-berlin.de/course/view.php?id=95156`

Exercise 9.1. (3 + 3 points) Let R be a ring.

1. Recall that an element $x \in R$ is nilpotent if $x^n = 0$ for some $n \geqslant 1$. Assume that for any prime ideal $\mathfrak{p} \subset R$ there exist no non-zero nilpotent elements in $R_{\mathfrak{p}}$. Prove that R contains no non-zero nilpotent elements. Hint: assuming that a non-zero $x \in R$ is nilpotent, consider its annihilator $\mathrm{Ann}(x) = \{y \in R \mid xy = 0\}$, prove that $\mathrm{Ann}(x)$ is contained in some prime ideal, and arrive at a contradiction.

2. Assume that for any prime ideal $\mathfrak{p} \subset R$ the ring $R_{\mathfrak{p}}$ is an integral domain. Is it true that R is an integral domain?

Exercise 9.2. (4 points) Let K be a field, and let $\nu\colon K^\times \to \mathbb{Z}$ be a surjective group homomorphism such that $\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$. Let $R = \{x \in K^\times \mid \nu(x) \geqslant 0\} \sqcup \{0\}$. Prove that R is a discrete valuation ring with valuation ν.

Exercise 9.3. (3 points) Let R be a Dedekind domain and $S \subset R$ a multiplicatively closed subset that does not contain zero. Prove that $S^{-1}R$ is a Dedekind domain.

Exercise 9.4. (3 points) Let $R = \mathbb{C}[[X]] = \{\sum_{i \geqslant 0} a_i X^i \mid a_i \in \mathbb{C}\}$ be the ring of formal power series with complex coefficients. Prove that R is a discrete valuation ring, describe the valuation, the maximal ideal, and the field of fractions of this ring.