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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 1

Aufgabe 1. Every ring has at least two different ideals. Characterize those rings that have precisely two ideals. Show that a ring that is not a field contains a principal ideal $\neq (0), (1)$.

Aufgabe 2. Use the corresponding facts from group theory to prove the following assertions:

i) Let $M_1 \subset M_2 \subset M$ be submodules. Then there exists a natural isomorphisms:

$$(M/M_1)/(M_2/M_1) \cong M/M_2.$$

ii) Let $M_1, M_2 \subset M$ be submodules. Then there exists a natural isomorphism:

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$

Aufgabe 3. Let M be an A-module. Show that there exists a natural isomorphism

$$M \cong \mathrm{Hom}_A(A, M).$$

Think of an example of an A-module $M \neq 0$ with $\mathrm{Hom}_A(M, A) = 0$.

Aufgabe 4. Let $f \colon M \to N$ be an A-module homomorphism.

i) Show that the kernel of the map $N \to \mathrm{Coker}(f)$ is naturally isomorphic to the image of f. (In abstract categorical language, this is the definition of the image.)

ii) Show that the cokernel of the map $\mathrm{Ker}(f) \to M$ is naturally isomorphic to the image of f. (This is what abstractly is called the coimage. So, for A-module homomorphisms, coimage and image coincide.)

Aufgabe 5. In class we have seen that $\mathrm{Hom}(\ , M)$ is left exact (contravariant). Show that $\mathrm{Hom}(M,\ )$ is also left exact (but covariant).

Aufgabe 6. Recall the definition of the tensor product $V \otimes_k W$ of vector spaces over a field k and its universal property. Generalize the universal property to the case of modules and show, just using this property, that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong 0$ for m, n coprime.

Please turn over.

Do not hand solutions for this sheet. It only serves as basis for the tutorials in week 2.

Exercises will be handed out each Monday or can be downloaded from:
`http://www.math.uni-bonn.de/people/aosoldat/commalg_V3A1_SS15.htmpl`
Solutions have to be handed in the following Monday before(!) the lecture.
So the first solutions are due on April 20.

Exams: first date - 29/07/2015; second date - 26/09/2015.