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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 11

Exercise 58. (4 points) Let A be an integral domain and $M \subset K = Q(A)$ a fractional ideal which is finitely generated. Show that for any prime ideal $\mathfrak{p} \subset A$, there exists an isomorphism of $A_{\mathfrak{p}}$-modules:

$$(A : M)_{\mathfrak{p}} \cong (A_{\mathfrak{p}} : M_{\mathfrak{p}}).$$

Exercise 59. (6 points) For any Noetherian ring A one defines $\mathrm{Pic}(A)$ as the set of all isomorphism classes of finite A-modules M such that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

i) Show that $\mathrm{Pic}(A)$ with $(M, N) \mapsto M \otimes_A N$ is an abelian group. (The *Picard group.*)

ii) Show that for a Dedekind ring the map $\mathrm{Cl}(A) \to \mathrm{Pic}(A)$ that forgets the inclusion $M \subset K = Q(A)$ of a fractional ideal is an isomorphism.

Exercise 60. (2 points)
i) Show that $A \coloneqq k[x_1, x_2]$ is not a Dedekind ring by describing a non-zero fractional ideal that is not invertible.
ii) We know that $A = k[x_1, x_2]/(x_2^2 - x_1^3)$ is not normal and hence not a Dedekind ring. Find a non-zero fractional ideal that is not invertible.

Exercise 61. (4 points)
Let A be a Dedekind ring and $S \subset A$ a multiplicative set.

i) Show that $S^{-1}A$ is again a Dedekind ring or the quotient field $Q(A)$.

ii) Show that $\mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$, $M \mapsto S^{-1}M$, defines a surjective group homomorphism.

Exercise 62. (3 points)
Prove that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$ have class number one, i.e.

$$h_i = 1 \text{ and } h_{\sqrt{-2}} = 1.$$

Hint: in the second example you may use without proof the fact that a prime number $p > 2$ is representable in the form $a^2 + 2b^2$ with $a, b \in \mathbb{Z}$ if and only if $p \equiv 1 \pmod 8$ or $p \equiv 3 \pmod 8$.

Due Monday Jun 29.