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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 13

Exercise 68.

i) Consider as usual a ring A as a module over itself. Prove that A is Artinian if and only if it admits a finite composition series.

ii) Prove that a ring A is Artinian if and only if it is Noetherian and $\dim A = 0$. Use this to show that for a Noetherian ring A and a maximal ideal $\mathfrak{m} \subset A$ every quotient ring $A/\mathfrak{m}^k$ is Artinian.

Exercise 69.
Consider the local ring $(A, \mathfrak{m})$ which is obtained as the localization of the ring $k[x, y]/(x^2 - y^3)$ (the cusp) resp. $k[x, y]/(y^2 - x^2(x + 1))$ (the node) in the maximal ideal (x, y). Determine in each case an $\mathfrak{m}$-primary principal ideal.

Exercise 70.
Let X be a topological space. Consider a chain $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$ of irreducible closed subsets of X. The integer n is called the length of the chain, and the *(Krull) dimension* of X, denoted $\dim X$, is the supremum of lengths of all such chains. Show that $\dim(A) = \dim \operatorname{Spec}(A)$ and $\operatorname{ht}(\mathfrak{p}) + \dim V(\mathfrak{p}) \leq \dim(A)$.

Exercise 71.
A polynomial $P \in \mathbb{Q}[T]$ is called *numerical* if $P(n) \in \mathbb{Z}$ for all $n \gg 0$. Prove the following assertions:

i) If $P \in \mathbb{Q}[T]$ is a numerical polynomial of degree r, then there exist $c_0, \ldots, c_r \in \mathbb{Z}$ such that
$$P(T) = c_0 \binom{T}{r} + c_1 \binom{T}{r-1} + \ldots + c_r,$$
where $\binom{T}{k} = \frac{T(T-1)\ldots(T-k+1)}{k!}$.

ii) Assume $f \colon \mathbb{Z} \to \mathbb{Z}$ is such that the induced *difference function*
$$\Delta f \colon \mathbb{Z} \to \mathbb{Z},\ n \mapsto f(n+1) - f(n)$$
is polynomial-like, i.e. there exists a (numerical) polynomial $Q \in \mathbb{Q}[T]$ with $\Delta f(n) = Q(n)$ for $n \gg 0$. Show that then also f is polynomial-like, i.e. there exists a (numerical) polynomial $P \in \mathbb{Q}[T]$ with $f(n) = P(n)$ for $n \gg 0$. Moreover, $\deg P(T) = \deg Q(T) + 1$.

Please turn over

Exams:
first date - 29/07/2015, 9-11, Wolfgang-Paul-Hörsaal, Kreuzbergweg 28;
second date - 26/09/2015, 9-11, Großer Hörsaal, Wegelerstraße 10.