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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 4

Exercise 20. (6 points)
Let A be a ring and $\mathfrak{a} \subset A$ an ideal. Prove the following consequences of the Nakayama lemma:

i) Let $N \to M$ be a homomorphism of A-modules such that the induced homomorphism $N/\mathfrak{a}N \to M/\mathfrak{a}M$ is surjective. If M is a finite A-module and $\mathfrak{a} \subset \mathfrak{R}$ where $\mathfrak{R}$ is the Jacobson radical, then $N \to M$ is surjective.

ii) Let $N \to M$ be a homomorphism of A-modules such that the induced homomorphism $N/\mathfrak{a}N \to M/\mathfrak{a}M$ is surjective. If M is a finite A-module, then there exists an element of the form $b = 1 + a$ with $a \in \mathfrak{a}$, such that the induced homomorphism $N_b \to M_b$ of A_b-modules is surjective.

iii) Assume that $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{a}M$. If M is finite, then there exists an element $b = 1 + a$ with $a \in \mathfrak{a}$, such that $m_1, \ldots, m_n$ generate the A_b-module M_b.

Exercise 21. (5 points)
Let M be a finite module over a local integral domain $(A, \mathfrak{m})$. Let $k \coloneqq A/\mathfrak{m}$ be its residue field and $K \coloneqq Q(A)$ its fraction field. Consider the k-vector space $M \otimes_A k$ and the K-vector space $M \otimes_A K$. Show that

$$\dim_K(M \otimes_A K) \leq \dim_k(M \otimes_A k)$$

and that equality holds if and only if M is free.

Exercise 22. (4 points)
Let k be a field and $A = k[X_1, X_2]/(X_2^2)$. Show that $S \coloneqq \{f(X_1) + X_2 \cdot g(X_1) \mid f(X_1) \neq 0\}$ is a multiplicative set and prove that there exists an isomorphism of rings

$$S^{-1}A \cong k(X_1)[X_2]/(X_2^2).$$

Exercise 23. (4 points)
Recall that $M/\mathfrak{a}M \cong M \otimes A/\mathfrak{a}$ for any A-module M and any ideal $\mathfrak{a} \subset A$. Use Nakayama lemma to show that for a local ring $(A, \mathfrak{m})$ the tensor product $M \otimes_A N$ of two finite A-modules M and N is trivial if and only if $M = 0$ or $N = 0$.

Exercise 24. (4 points)
Let $a \in A \setminus \mathfrak{N}$ (where $\mathfrak{N}$ is the nilradical) and $f \colon A \to A_a$ be the natural ring homomorphism. Show that the induced map $\varphi \colon \operatorname{Spec}(A_a) \to \operatorname{Spec}(A)$ induces a homeomorphism $\psi \colon \operatorname{Spec}(A_a) \xrightarrow{\sim} D(a)$ (i.e. ψ and ψ^{-1} are bijective and continuous).

Due Monday May 4.