Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 5

Exercise 25. (2 points)
Let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset A$ be two prime ideals. Show that the localization of $A_{\mathfrak{p}_2}$ in the prime ideal corresponding to $\mathfrak{p}_1$ is isomorphic to $A_{\mathfrak{p}_1}$.

Exercise 26. (2 points)
Show that for any A-module M which is finitely presented, i.e. there exists an exact sequence $A^{\oplus n} \to A^{\oplus m} \to M \to 0$, any prime ideal $\mathfrak{p} \subset A$ and any A-module N the natural homomorphism

$$\mathrm{Hom}_A(M, N)_{\mathfrak{p}} \to \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

is an isomorphism.

Exercise 27. (3 points)
Let A be a ring and $a, b \in A \setminus \mathfrak{N}$. Show that $D(a) \subset D(b)$ if and only if $\frac{b}{1} \in A_a$ is a unit. Furthermore, show that in this case the natural ring homomorphism $A \to A_a$ factorizes via a ring homomorphism $A_b \to A_a$. Conclude from this that $D(a) = D(b)$ if and only if $A_a \cong A_b$.

Exercise 28. (6 points)

i) Recall from Exercise 12 the definitions of directed systems of modules and direct limits of such systems. Given a partially orded directed set I we can consider directed systems of rings $(A_i)_{i \in I}$, where the maps $f_{ij} \colon A_i \to A_j$ are assumed to be ring homomorphisms. Every ring is naturally a $\mathbb{Z}$-module, so there exists a direct limit $\varinjlim A_j$ in the category of $\mathbb{Z}$-modules. Show that $\varinjlim A_j$ carries a natural ring structure and the morphisms $A_i \to \varinjlim A_j$ are ring homomorphisms.

ii) Make the set of open subsets of $\mathrm{Spec}(A)$ into a partially ordered directed set by defining $U \leq V$ if $V \subset U$. Recall that for $\mathfrak{p} \in \mathrm{Spec}(A)$ we have $\mathrm{Spec}(A_{\mathfrak{p}}) = \bigcap_{\mathfrak{p} \in U} U$ and show that there exists a natural isomorphism

$$A_{\mathfrak{p}} \cong \varinjlim A_a,$$

where the direct limit is over all $a \in A$ with $\mathfrak{p} \in D(a)$. (The localizations A_a form a directed system of rings due to Exercise 27.)

Please turn over

Due Monday May 11.

Exercise 29. (6 points)
A *graded ring* is a ring A that can be written as $A = \bigoplus_{n=0}^{\infty} A_n$ such that $A_n \subset A$ are additive subgroups and the multiplication satisfies $A_m \cdot A_n \subset A_{m+n}$. The polynomial ring $A = k[X_1, \dots, X_N]$ with A_n the set of homogeneous polynomial of total degree n is an example.

i) An ideal $\mathfrak{a} \subset A$ is called homogeneous if $\mathfrak{a} = \bigoplus (A_n \cap \mathfrak{a})$. Show that $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cdot \mathfrak{b}$ are homogeneous if $\mathfrak{a}, \mathfrak{b} \subset A$ are.

ii) Show that a homogeneous ideal $\mathfrak{p} \subset A$ is a prime ideal if for all $a \in A_m$ and $b \in A_n$ with $a \cdot b \in \mathfrak{p}$ one has $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

iii) Define $\mathrm{Proj}(A) \coloneqq \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ homogeneous prime ideal with } A_+ \coloneqq \bigoplus_{n>0} A_n \not\subset \mathfrak{p}\}$ and show that $\mathrm{Proj}(k[X]) = \{(0)\}$ and $\mathrm{Proj}(k[X_0, X_1]) = \{(0), (X_0), (\lambda X_0 - X_1) \mid \lambda \in k\}$ if $k = \bar{k}$.