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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 8

Exercise 42. (2 points)
Let A be a subring of B such that B is integral over A. Show that every ring homomorphism $f\colon A \to K$ with K an algebraically closed field can be extended to a ring homomorphism $\tilde{f}\colon B \to K$.

Exercise 43. (2 points)
Let A be a subring of B and assume that $S \coloneqq B \setminus A$ is closed under multiplication. Show that then A is integrally closed in B.

Exercise 44. (4 points)
i) Show that the $\mathbb{C}$-algebra homomorphism $A \coloneqq \mathbb{C}[X,Y]/(X^3 - Y^2) \to \mathbb{C}[T]$, $X \mapsto T^2$, $Y \mapsto T^3$ induces a bijection $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[T]) \to \mathrm{Spec}(A)$, but that there does not exist a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[T] \to A$ that would induce a bijection $\mathrm{Spec}(A) \to \mathbb{A}^1_{\mathbb{C}}$

ii) Discuss in a similar way $A \coloneqq \mathbb{Z}[x]/(x^2+4) \to \mathbb{Z}[i]$, $x \mapsto 2i$.

Exercise 45. (4 points)
Let A be a ring and G a finite group of ring automorphisms $g\colon A \xrightarrow{\sim} A$.

i) Show that A is integral over the *ring of G-invariants* $A^G \coloneqq \{a \in A \mid g(a) = a \text{ for all } g \in G\}$.

ii) Let $\mathfrak{p} \subset A^G$ be a prime ideal. Prove that G acts transitively on the set of those prime ideals $\mathfrak{q} \subset A$ for which $\mathfrak{q} \cap A^G = \mathfrak{p}$.

Exercise 46. (3 points)
Let A be a normal ring and let K be its quotient field. Show that for a finite Galois extenions L/K the integral closure $\bar{A}$ of A in L is invariant under $G \coloneqq \mathrm{Gal}(L/K)$, i.e. for all $g \in G$ one has $g(\bar{A}) = \bar{A}$, and that $\bar{A}^G = A$.

Exercise 47. (4 points)
Consider the ring $A \coloneqq k[x,y]/(x^2+y^2-1)$. Show that it is factorial for $k = \mathbb{C}$ and not factorial for $k = \mathbb{R}$. (Observe that in the two cases A is isomorphic to the ring of functions $\mathbb{C}[e^{it}, e^{-it}]$ resp. $\mathbb{R}[\sin(t), \cos(t)]$.)

Due Monday Jun 8.

Information from the student council: The student council of mathematics will organize the math party on 11/06 in N8schicht. The presale will be held on Mon 8/06, Tue 9/06 and Wed 10/06 in front of mensa Poppelsdorf. Further information can be found at www.fsmath.uni-bonn.de