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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 3

Exercise 12. (4 points) *Direct and inverse image are adjoint.*
Let $f\colon X \to Y$ be a continuous map. Show that $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is right adjoint to $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (one writes $f^{-1} \dashv f_*$), i.e. for all $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$, there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Show that in particular there exist natural homomorphisms

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \text{ and } f^{-1}f_*\mathcal{F} \to \mathcal{F}.$$

Verify also that for the composition of two continuous maps $f : X \to Y$ and $g : Y \to Z$ one has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 13. (2 points) *Local rings of continuous functions.*
Let X be a topological space and let $\mathcal{C}$ be the sheaf of continuous functions on X. Consider for a point $x \in X$ the stalk $\mathcal{C}_x$. Show that $\mathcal{C}_x \to \mathbb{R}$, $f \mapsto f(x)$ is a well defined map and that $\mathcal{C}_x$ is a local ring with maximal ideal $\mathfrak{m}_x := \{f \in \mathcal{C}_x \mid f(x) = 0\}$. Describe similar situations involving differentiable or holomorphic functions.

Exercise 14. (3 points) *Direct image is left exact.*
Let $f\colon X \to Y$ be a continuous map. Prove that $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is left exact. Give two different proofs: first using only the definition of f_*, and second using the fact that f_* is right adjoint (see exercise 12).

Exercise 15. (4 points) *Direct sum and sheaf $\mathcal{H}om$.*
Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of abelian groups on a topological space X

i) Show that $\mathcal{F} \oplus \mathcal{G}\colon U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ defines a sheaf. As an example, let $\mathcal{F} = \mathcal{G}$ be the sheaf of continuous functions with values in $\mathbb{R}$. Observe that $\mathcal{F} \oplus \mathcal{G}$ is the sheaf of continuous functions with values in $\mathbb{R}^2$.

ii) For any open set $U \subset X$ the set of morphisms $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an abelian group. Show that
$$U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$
defines a sheaf of abelian groups on X. It will be denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Exercise 16. (4 points) *Gluing of sheaves.*
Let X be a topological space and let $X = \bigcup U_i$ be an open covering. We use the shorthand $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Consider sheaves $\mathcal{F}_i$ on U_i and *gluings* $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$, where $\varphi_{ii} = \mathrm{id}$ for any i. Show that if the *cocycle condition* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk} is satisified, then there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\varphi_i : \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on U_{ij}. The $(\mathcal{F}, \varphi_i)$ is unique up to unique isomorphism.

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Due Monday 9 November, 2015. Before(!!) the lecture.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 17. (4 extra points) *Functor of points and the Yoneda lemma*
Let $\mathcal{C}$ be a category[1] with sets of morphisms between two objects X, Y denoted $\mathrm{Hom}(X, Y)$. Then every object X in $\mathcal{C}$ induces a functor

$$h_X\colon \mathcal{C}^{\mathrm{op}} \to (\mathit{Sets}),\ Y \mapsto \mathrm{Hom}(Y, X).$$

Observe that $h_X(X)$ contains a distinguished element.

i) Consider the three categories $\mathcal{C} := (\mathit{Top})$ (of topological spaces); $\mathcal{C} := (\mathit{Ab})$ (of abelian groups); $\mathcal{C} := (\mathit{Rings})$ (of rings with unit) and denote for each object X in $\mathcal{C}$ by $|X|$ the underlying set. Show that in all three cases there exists an object Z in $\mathcal{C}$ such that $|X|$ can be recovered as $|X| = h_X(Z)$.

ii) Consider the category of affine schemes $\mathcal{C} := (\mathit{AffSch})$. Does there exist an object as in i) in this case?

iii) For an arbitrary category $\mathcal{C}$, denote by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets}))$ the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathit{Sets})$ and view $X \mapsto h_X$ as a functor

$$h\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets})).$$

The Yoneda lemma then asserts that h is a fully faithful embedding, in other words h defines an equivalence of categories between $\mathcal{C}$ and a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathit{Sets}))$. Spell out what this means and try to prove it. Check Vakil's notes on the subject (or any other source). Objects in the image of h (or, more precisely, objects isomorphic to objects in the image) are called *representable functors*.

[1] All set theoretic issues (e.g. whether the category is small or not) will be ignored