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Summer term 2016

Exercises, Algebraic Geometry II – Week 10

Exercise 45. (2 points) *Morphisms between curves.*
Let $f: C \to D$ be a finite morphism between reduced projective curves over a field k. Show that f is an isomorphism if and only if the natural map $\mathcal{O}_D \to f_*\mathcal{O}_C$ is an isomorphism. Compare this to Hurwitz formula.

Exercise 46. (3 points) *Local analytic invariants.*
Let C be an integral curve over a field k and let $f: \tilde{C} \to C$ be its normalization. For a closed point $x \in C$ define $\delta_x := \text{length}(\tilde{\mathcal{O}}_{C,x}/\mathcal{O}_{C,x})$, where $\mathcal{O}_{C,x} \subset \tilde{\mathcal{O}}_{C,x} \subset K(C)$ is the normalization. Assume that for two curves (C,x) and (C',x') we have $\widehat{\mathcal{O}}_{C,x} \cong \widehat{\mathcal{O}}_{C',x'}$, and the normalized rings $\tilde{\mathcal{O}}_{C,x}$ and $\tilde{\mathcal{O}}_{C',x'}$ are DVR. Show that one has $\delta_x = \delta_{x'}$.

Exercise 47. (3 points) *Testing isomorphisms on local rings.*
Let $f: X \to Y$ be a morphism of finite type with Y locally Noetherian. Suppose that for a point $y \in Y$ the induced morphism $X \times_Y \text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism. Show that then there exists an open neighbourhood $y \in U \subset Y$ such that the induced morphism $X \times_Y U \to U$ is an isomorphism. (See [Liu, Ex. 3.2.5] for a relative version of this statement.)

Exercise 48. (3 points) *Connectedness under base change.*
Consider a morphism $f: X \to Y$.

1. Show that the property $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ is stable under flat base change.
2. Show that the property $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ is not necessarily stable under arbitrary base change.
3. Show that having connected fibres is not even stable under flat base change.

Exercise 49. (2 points) *(Dis)connected fibres.*
Find examples of morphisms $f: X \to Y$ between integral schemes with $\mathcal{O}_Y \not\cong f_*\mathcal{O}_X$ with all fibres being (geometrically) connected resp. all fibres being disconnected.

Due Monday 27 June, 2016.