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Summer term 2016

Exercises, Algebraic Geometry II – Week 12

Exercise 50. (2 points) *Connectedness.*
Let $f\colon X \to Y$ be a projective morphism of Noetherian schemes. Assume that f is surjective, Y is connected and all the fibres of f are connected. Show that X is also connected. Replace 'connected' by 'irreducible'. Is the assertion still true?

Exercise 51. (4 points) *Rigidity I.*
Let $f\colon X \to Y$ and $g\colon X \to Z$ be projective morphisms of varieties (i.e. integral schemes of finite type over a field) with $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ and such that g contracts each fibre of f (i.e. $g(f^{-1}(y))$ is a point for each $y \in Y$). Show that there exists a morphism $h\colon Y \to Z$ with $h \circ f = g$.

Exercise 52. (4 points) *Rigidity II.*
Let X, Y and Z be varieties over an algebraically closed field k, and let X be proper over k. Let $f\colon X \times Y \to Z$ be a morphism. Assume that there exists a closed point $y_0 \in Y$, such that $f(X \times \{y_0\})$ is a single point in Z. Prove that there exists $g\colon Y \to Z$, such that $f = g \circ \pi_Y$ where $\pi_Y\colon X \times Y \to Y$ is the projection.

Exercise 53. (2 points) *Adjunction.*
Let $f\colon X \to Y$ be a morphism of schemes with $\mathcal{O}_Y \xrightarrow{\sim} f_*\mathcal{O}_X$. Show that for every invertible sheaf $\mathcal{L}$ on Y the natural adjunction morphism $\mathcal{L} \to f_*f^*\mathcal{L}$ is an isomorphism. Suppose $\mathcal{L}$ is an invertible sheaf on X. What can you say about the adjunction morphism $f^*f_*\mathcal{L} \to \mathcal{L}$?

Exercise 54. (4 points) *Globally generated line bundles.*
Let $\mathcal{L}$ be a globally generated invertible sheaf on a normal projective scheme X over a field k. Consider the induced morphism $\varphi\colon X \to \mathbb{P}^N_k$. Show that the morphism φ can be decomposed as $\varphi = g \circ \varphi'$ with $\varphi'\colon X \to Z$ projective with connected fibres and $g\colon Z \to \mathbb{P}^N_k$ finite such that:

1. $\deg(\mathcal{L}|_C) = 0$ for a complete connected curve $C \subset X$ if and only if $\varphi(C) = \mathrm{pt}$.
2. Z is normal.

Assume that for every complete curve $C \subset X$ we have $\deg(\mathcal{L}|_C) \neq 0$. Show that $\mathcal{L}$ is ample.

Exercise 55. (2 points) *Components of the fibre.*
Let $f\colon X \to Y$ be a projective morphism with Y locally Noetherian. Show that connected components of a fibre X_y are in bijection with maximal ideals of $(f_*\mathcal{O}_X)_y$.

Due Monday 11 July, 2016.