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Summer term 2016

Exercises, Algebraic Geometry II – Week 2

Exercise 6. (6 points) *Conics.*
Let $C \subset \mathbb{P}^2_k$ be a geometrically integral plane curve defined by a quadratic equation (a 'smooth conic'). Assume that $\mathrm{char}(k) \neq 2$.

1. Show that $C \cong \mathbb{P}^1_k$ if and only if $C(k) \neq \emptyset$.
2. Find one example which is not rational (i.e. $C \ncong \mathbb{P}^1_k$).
3. Prove that there exists a quadratic field extension $k \subset K$, such that $C \times_k \mathrm{Spec}(K) \cong \mathbb{P}^1_K$.

Exercise 7. (6 points) *Unirational conic bundle.*
A variety X is unirational if there exists a dominant rational map $\mathbb{P}^n \to X$ for some n. A rational conic bundle is a morphism between varieties $\varphi : X \to S$ such that there exists an open set $\emptyset \neq U \subset S$ and an embedding $X_U := X \times_S U \hookrightarrow \mathbb{P}^2_U$ whose fibres are conics in $\mathbb{P}^2$. Assume $T \subset X$ is a subvariety, such that the morphism $\varphi|_T : T \to S$ is dominant. Prove the following assertions.

1. The generic fibre of φ is irreducible.
2. If the generic fibre is a smooth conic (see the previous exercise) over $K(S)$, the morphism $\phi|_T$ is birational and S is rational, then X is rational (i.e. birational to $\mathbb{P}^n$).
3. If the generic fibre is a smooth conic over $K(S)$ and T is unirational, then X is unirational.

Exercise 8. (4 points) *Curves and function fields.*
Let k be a field of characteristic zero. Consider the two function fields $K_1 := k(x_1, y_1)$ and $K_2 := k(x_2, y_2)$ with $y_1^2 = x_1^3 + 4x_1^2 + 3x_1$ and $y_2^2 + x_2^4 + 1 = 0$, respectively. Show that K_1 and K_2 are not isomorphic (you may assume without a proof that the corresponding projective curves are normal).

Exercise 9. (4 points) *Blowing up the vertex.*
Let $X \subset \mathbb{P}^2_k$ be a curve defined by a homogeneous polynomial f. Let Y be the cone over X, that is the subvariety in $\mathbb{A}^3_k$ defined by the same polynomial f. Let $\varphi : \tilde{Y} \to Y$ be the blow-up of Y at the point $P = (0, 0, 0)$. Prove that $\varphi^{-1}(P) \simeq X$.

Due Monday 25 April, 2016. Before the lecture.