Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2016

Exercises, Algebraic Geometry II – Week 3

Exercise 10. (3 points) *Cotangent sheaf of a product.*
Let X and Y be schemes over S. Show that $\Omega_{X\times_S Y/S}$ is isomorphic to the direct sum of the pull-backs of $\Omega_{X/S}$ and $\Omega_{Y/S}$, i.e.

$$\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}.$$

Exercise 11. (3 points) *Relative Euler sequence.*
Let $\mathcal{E}$ be a locally free sheaf on a scheme Y and let $\pi\colon X := \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(S^*(\mathcal{E})) \to Y$ be the associated projective bundle. Show that there exists a natural exact sequence

$$0 \to \Omega_{X/Y} \to \pi^*\mathcal{E} \otimes \mathcal{O}_\pi(-1) \to \mathcal{O}_X \to 0.$$

Here, $\mathcal{O}_\pi(-1)$ is the invertible sheaf associated with $S^*(\mathcal{E})(-1)$.

Exercise 12. (6 points) *The Jouanolou trick.*
Prologue: Let X be a projective variety over a field k (algebraically closed for simplicity). Is it feasible that there exists a surjective morphism $f\colon Y \to X$ with Y affine and all fibres isomorphic to affine spaces $\mathbb{A}^n_k$ (of constant dimension)? Think about this question for ten minutes before doing the following exercise.

1. Let V be a vector space of dimension $n+1$ and V^* its dual. We write $\mathbb{P}(V) := \mathrm{Proj}(S^*(V^*))$ and $\mathbb{P}(V^*) = \mathrm{Proj}(S^*(V))$ (and think of them as the projective space of lines $\ell \subset V$ resp. hyperplanes $H \subset V$). Consider the 'incidence variety'
$$\Gamma := \{(\ell, H) \mid \ell \subset H\} \subset \mathbb{P}(V) \times \mathbb{P}(V^*),$$
which is defined by the equation obtained from the dual pairing $V \times V^* \to k$.
2. Use the Segre embedding to show that $Y := \mathbb{P}(V) \times \mathbb{P}(V^*) \setminus \Gamma$ is affine.
3. Show that the fibres of the first projection $\pi\colon Y \to \mathbb{P}(V)$ are isomorphic to affine spaces $\mathbb{A}^n$.
4. Show that there exists an open covering $\mathbb{P}(V) = \bigcup U_i$ and isomorphisms $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^n$ compatible with the projections.
5. Use the above to prove the following statement: For any projective variety X there exists an affine variety Y and a morphism $\pi\colon Y \to X$ which is a Zariski locally trivial $\mathbb{A}^n$-bundle.

Epilogue: So, by passing to an $\mathbb{A}^n$-bundle, any projective variety X becomes affine. The construction can be performed over $\mathrm{Spec}(\mathbb{Z})$. Moreover, $Y \to X$ as above exists for any scheme X smooth over $\mathrm{Spec}(A)$ with A a Noetherian and regular ring (Thomason's extension). Topologists phrase this result as: 'Up to $\mathbb{A}^1$-weak equivalence, any smooth A-scheme is an affine scheme smooth over A'.

Please turn over

Due Monday 2 May, 2016. Before the lecture.

Exercise 13. (4 points) *The Jouanolou trick: Matrix version.*
Let k be an algebraically closed field. For $n \geq 1$ consider the set Y of all matrices $A \in M(n+1, n+1, k)$ of rank one satisfying $A^2 = A$.

1. Show that Y is naturally an affine variety.
2. Show that the fibres of the morphism $\pi \colon Y \to \mathbb{P}^n_k$, $A \mapsto \mathrm{Im}(A)$ are isomorphic to $\mathbb{A}^n_k$.
3. Compare this construction with the one in the previous exercise.

Exercise 14. (4 points) *Plurigenera of smooth plane curves.*
Consider a smooth curve $C \subset \mathbb{P}^2_k$ defined by a polynomial of degree d.

1. Show that $\Omega_{C/k} \cong \mathcal{O}(d-3)|_C$.
2. Compute $h^0(C, \Omega_{C/k}^{\otimes n}) \coloneqq \dim H^0(C, \Omega_{C/k}^{\otimes n})$.
3. Compare $h^0(C, \Omega_{C/k})$ with the arithmetic genus of C.

The student council of mathematics will organize the math party on 12/05 in N8schicht. The presale will be held on Mon 09/05, Tue 10/05 and Wed 11/05 in front of mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de