Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2016

Exercises, Algebraic Geometry II – Week 8

Exercise 36. (3 points) *Composition of étale and unramified morphisms.*
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms such that $g \circ f$ is étale and g is unramified. Show that then also f is étale.

Exercise 37. (3 points) *Étale covering of a nodal cubic.*
Let Y be the plane nodal cubic curve $y^2 = x^2(x+1)$. Show that Y has finite étale (or at least unramified) covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Hint: you can either use the next exercise, or consider the map $\mathbb{A}^2 \to \mathbb{A}^2$, $(s,t) \mapsto (s^2-1, st)$). Draw a picture.

Exercise 38. (4 points) *Cyclic étale coverings.*
Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a two-torsion line bundle, i.e. $\mathcal{L}^2 \cong \mathcal{O}_X$, on a k-scheme X over a field k of characteristic $\neq 2$. Define an $\mathcal{O}_X$-algebra structure on $\mathcal{A} \coloneqq \mathcal{O}_X \oplus \mathcal{L}$ by $(a \oplus b)\cdot(a' \oplus b') = (aa' + \varphi(bb')) \oplus (ab' + a'b)$. Here $\varphi\colon \mathcal{L}^2 \xrightarrow{\sim} \mathcal{O}_X$ is a fixed trivialization and we use the standard multiplication $H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$, $a \otimes b \mapsto ab$.

Show that $Y \coloneqq \mathrm{Spec}(\mathcal{A}) \to X$ is étale of degree two. Furthermore, show that if X is smooth and irreducible then Y is irreducible if and only if $\mathcal{L}$ is not trivial.

Can this be generalized to n-torsion line bundles, i.e. those with $\mathcal{L}^n \cong \mathcal{O}_X$?

Exercise 39. (3 points) *Taking roots of sections.*
Let X be a smooth variety over a field k. Fix a section $0 \neq s \in H^0(X, \mathcal{L}^n)$, where $\mathcal{L}$ is a line bundle on X. Show that there exists a finite surjective morphism $\pi\colon Y \to X$ and a section $t \in H^0(Y, \pi^*\mathcal{L})$ with $\pi^* s = t^n$. For this, let $\mathbb{V}(\mathcal{L}^*) \coloneqq \mathrm{Spec}(\bigoplus_{i \geq 0} (\mathcal{L}^*)^i)$ be the vector bundle associated with $\mathcal{L}^*$ (see last semester). Let $\tilde{\pi}\colon \mathbb{V}(\mathcal{L}^*) \to X$ be the projection and define $Y = Z(\tilde{\pi}^* s - \tilde{t}^n)$, where $\tilde{t} \in H^0(\mathbb{V}(\mathcal{L}^*), \tilde{\pi}^*\mathcal{L}) = H^0(X, \mathcal{L} \otimes \tilde{\pi}_* \mathcal{O}_{\mathbb{V}(\mathcal{L}^*)}) = H^0(X, \mathcal{L} \otimes \bigoplus_{i \leq 0} \mathcal{L}^i)$ is the natural trivializing section of $\mathcal{L} \otimes \mathcal{L}^*$. Then set $t \coloneqq \tilde{t}|_Y$ and $\pi = \tilde{\pi}|_Y$.

Assuming that $\mathrm{char}(k)$ and n are coprime, determine the ramification divisor of the projection $\pi\colon Y \to X$ and describe the canonical bundle of Y in terms of X and $\mathcal{L}$.

Due Monday 13 June, 2016.