Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2016

Exercises, Algebraic Geometry II – Week 9

Exercise 40. (3 points) *Associated graded rings.*
Let A be a Noetherian ring, $\mathfrak{a} \subset A$ an ideal and $\hat{A}$ the $\mathfrak{a}$-adic completion. Prove that $\hat{\mathfrak{a}}^k \simeq \mathfrak{a}^k \hat{A}$ for all $k \geq 0$. Deduce that $\mathrm{gr}_{\mathfrak{a}}(A) \simeq \mathrm{gr}_{\hat{\mathfrak{a}}}(\hat{A})$.

Exercise 41. (3 points) *Completion and localization/quotient.*
Let A be a Noetherian ring, $S \subset A$ a multiplicative system and $\mathfrak{a} \subset A$ an ideal. For an A-module M denote by $\widehat{M}$ the $\mathfrak{a}$-adic completion. Is it true that for M finitely generated $S^{-1}\widehat{M} \simeq \widehat{S^{-1}M}$? Let $\mathfrak{b} \subset A$ be another ideal. Is it true that $\hat{A}/\mathfrak{b}\hat{A} \simeq \widehat{(A/\mathfrak{b})}$?

Exercise 42. (4 points) *Flatness of the Frobenius.*
Recall the notion of the relative Frobenius $F_{X/S}$: Assume X is finite type over $\mathbb{F}_q$, $q = p^n$, and let $\pi \colon X \to S := \mathrm{Spec}(\mathbb{F}_q)$ be the structure morphism. Then the relative Frobenius $F_{X/S} \colon X \to X^{(p)} := X \times_{S,F} S$ (which is an S-morphism) is obtained from the universal property of the fibre product (with $F \colon S \to S$ given by the Frobenius $x \mapsto x^p$) applied to π and the absolute Frobenius $F \colon X \to X$.

Assume that X is smooth over $\mathbb{F}_q$ and show that then $F_{X/S} \colon X \to X^{(p)}$ is flat at all rational points of X.

Exercise 43. (2 points) *Complete local rings of singularities.*
Consider completions of the rings $k[x,y]/(xy)$ and $k[x,y]/(y^2 - x^4)$ at the maximal ideal (x,y) which corresponds to the singular points. Are these completions isomorphic?

Exercise 44. (3 points) *Complete local rings are Henselian.*
Let A be a local ring, $\mathfrak{m}$ its maximal ideal. Assume that A is $\mathfrak{m}$-adically complete. Let $f \in A[x]$ be a monic polynomial of degree n, and denote by $\bar{f} \in (A/\mathfrak{m})[x]$ its reduction mod $\mathfrak{m}$. Assume that there exist coprime monic polynomials $\bar{g}, \bar{h} \in (A/\mathfrak{m})[x]$ of degrees r, $n - r$ with $\bar{f} = \bar{g}\bar{h}$. Prove that one can find $g, h \in A[x]$, such that $g \equiv \bar{g} \,(\mathrm{mod}\, \mathfrak{m})$, $h \equiv \bar{h} \,(\mathrm{mod}\, \mathfrak{m})$ and $f = gh$.

Due Monday 20 June, 2016.