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Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 10

Exercise 42. *Blowing up the plane.*
Show that the Chow ring of the plane $\mathbb{P}^2$ blown-up in one point is described by

$$\mathrm{CH}^*(\mathrm{Bl}_x(\mathbb{P}^2)) \cong \mathbb{Z}[h,s]/(h^3, hs, s^2+h^2). \tag{1}$$

Recall that $\mathrm{Bl}_x(\mathbb{P}^2)$ can alternatively be described as the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Compare the induced descriptions of the Chow ring with (1).

Exercise 43. *The motive of a blow-up of $\mathbb{P}^3$ in a twisted cubic.*
Consider the blow-up $\mathrm{Bl}_Z(\mathbb{P}^3) \to \mathbb{P}^3$ in a twisted cubic $Z \subset \mathbb{P}^3$ (i.e. the image of $\mathbb{P}^1 \to \mathbb{P}^3$, $[t_0 : t_1] \mapsto [t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$) and denote the exceptional divisor by $j \colon E = \mathbb{P}(\mathcal{N}_{Z/\mathbb{P}^3}) \hookrightarrow \mathrm{Bl}_Z(\mathbb{P}^3)$.

(i) Show that the total Chern class of the normal bundle $\mathcal{N} := \mathcal{N}_{Z/\mathbb{P}^3}$ is given by $\mathrm{c}(\mathcal{N}) = 1 + 10h_0$, where h_0 is the class of a point on $Z = \mathbb{P}^1$. (This suggests that $\mathcal{N} \cong \mathcal{O}(5) \oplus \mathcal{O}(5)$. Is this actually true?)

(ii) Show that $\mathrm{CH}^*(E) \cong \mathbb{Z}[h_0, \xi]/(h_0^2, \xi^2 + 10h_0\xi)$.

(iii) Show that $\mathrm{CH}^1(\mathrm{Bl}_Z(\mathbb{P}^3)) \cong \mathbb{Z}h \oplus \mathbb{Z}[E]$, where h is the pull-back the class of a line.

(iv) Show that $\mathrm{CH}^2(\mathrm{Bl}_Z(\mathbb{P}^3))$ is generated by $h^2, j_*h_0, j_*\xi$ with one relation $3h^2 - 10j_*h_0 - j_*\xi$.

(v) Describe the multiplication $\mathrm{CH}^1 \times \mathrm{CH}^2 \to \mathrm{CH}^2$ and $\mathrm{CH}^1 \times \mathrm{CH}^2 \to \mathrm{CH}^3 \cong \mathbb{Z}$.

Exercise 44. *The motive of the blow-up.*
Use Manin's identity principle to prove that for the blow up $\tilde{X} := \mathrm{Bl}_Z(X)$ of a smooth projective variety X in a smooth subvariety $Z \subset X$ of codimension c there exists an isomorphism

$$\mathfrak{h}(\tilde{X}) \cong \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{c-1} \mathfrak{h}(Z)(-i).$$

Exercise 45. *Lefschetz standard conjecture.*
(i) Assume the Lefschetz standard conjecture $B(X, \mathcal{L})$ holds for the smooth projective variety X with an ample line bundle $\mathcal{L}$. Show that then $B(X \times X, \mathcal{L} \boxtimes \mathcal{L})$ holds, so in particular $B(C_1 \times C_2)$ holds for product of curves.

(ii) Assume $B(X)$ holds. Does this imply that also $B(\mathrm{Bl}_Z(X))$ holds (e.g. when Z is a point or a curve)?

Due Wednesday 11 January, 2017.

Exercise 46. *Chow groups of generic Lefschetz pencils.*
A Lefschetz pencil in $\mathbb{P}^n$ is a linear system of hypersurfaces X_t, $t \in \mathbb{P}^1$, where $\mathbb{P}^1$ is a line in the complete linear system $|\mathcal{O}(d)|$. More explicitly, $X_t = Z(f + tg)$ with two linearly independent $f, g \in H^0(\mathbb{P}^n, \mathcal{O}(d))$. For generic choice of f and g, the base locus of the liner system $B := Z(f, g)$ is smooth and the family of hypersurfaces $\{X_t\}$ can be described as the fibres of the morphism $\mathcal{X} := \mathrm{Bl}_B(\mathbb{P}^n) \to \mathbb{P}^1$, which resolves the indeterminacies of the rational map $\mathbb{P}^n \to \mathbb{P}^1$, $x \mapsto [f(x) : g(x)]$.

For simplicity we work over an algebraically closed field and assume $n = 3$.

(i) Compute $\mathrm{CH}^2(\mathcal{X})$.

(ii) Compute $\mathrm{CH}^2(\mathcal{X}_\eta)$ of the generic fibre $\mathcal{X}_\eta$ (over $k(\eta) = k(t)$).

(iii) Find examples where $\mathrm{CH}^2(\mathcal{X}_t)$ of a closed fibre $\mathcal{X}_t$ are 'bigger' than $\mathrm{CH}^2(\mathcal{X}_\eta)$.

Exercise 47. *Classes of proper transforms.*
Let $C_1, C_2 \subset \mathbb{P}^3$ be two distinct smooth curves and view the intersection $D := C_1 \cap C_2$ as a divisor on C_1. Let $X := \mathrm{Bl}_{C_1}(\mathbb{P}^3)$ and $\tilde{C}_2 \subset X$ the proper transform of C_2. Describe its class $[\tilde{C}_2] \in \mathrm{CH}^2(X)$.