Prof. Dr. Daniel Huybrechts
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Intersection theory and pure motives, Exercises – Week 6

Exercise 22. *Pseudo-abelian hull.*
Let $\mathcal{A}$ be an additive category, $\mathcal{B}$ a pseudo-abelian category and $\Phi\colon \mathcal{A} \to \mathcal{B}$ a fully faithful functor such that every object in $\mathcal{B}$ is a direct summand of an object in $\mathcal{A}$ (via Φ). Show that then $\mathcal{B}$ is equivalent to the pseudo-abelian hull $\tilde{\mathcal{A}}$ of $\mathcal{A}$.

Let $(A,p),(B,\mathrm{id})$ be objects in $\tilde{\mathcal{A}}$ and $\varphi\colon (A,p) \to (B,\mathrm{id})$ and $\psi\colon (B,\mathrm{id}) \to (A,p)$ be morphisms in $\tilde{\mathcal{A}}$ with $\varphi \circ \psi = \mathrm{id}_{(B,\mathrm{id})}$. Show that $p - \psi \circ \varphi$ is a projector for A and that $(A,p) \cong (B,\mathrm{id}) \oplus (A, p - \psi \circ \varphi)$.

Exercise 23. *Top degree motive.*
Recall the definition of $\mathfrak{h}^{2d}(X) \in \mathrm{Mot}(k)$ for a smooth projective variety X of dimension d and that it *a priori* depends on the choice of a closed point $x \in X$. Prove that

$$\mathfrak{h}^{2d}(X) \cong (k, \mathrm{id}, -d) \cong \mathbb{L}^d,$$

which in particular shows that $\mathfrak{h}^{2d}(X)$ does not depend on x.

Exercise 24. *Composition in* $\mathrm{Mot}(k)$.
Recall that $\mathrm{Mor}_{\mathrm{Mot}}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{n-m}(X,Y)p$ and convince yourself that convolution defines a $\mathbb{Q}$-linear associative composition on $\mathrm{Mot}(k)$.

Exercise 25. *Examples of Schubert cycles.*
We study some classical cycles on the Grassmannian $\mathrm{G}(2,4) = \mathbb{G}(1,3)$ of planes in k^4 assuming $k = \bar{k}$. For this fix (generic) subspaces $\ell_0, H_0, W_0 \subset k^4$ of dimension $1, 2$, and 3, respectively. Study the subschemes (with the induced reduced structure) $\Sigma_{2,1} \coloneqq \{H \mid \ell_0 \subset H \subset W_0\}$, $\Sigma_{1,0} \coloneqq \{H \mid H \cap H_0 \neq 0\}$, and $\Sigma_{1,1} \coloneqq \{H \mid H \subset W_0\}$ and their classes $\sigma_{i,j} \coloneqq [\Sigma_{i,j}] \in \mathrm{CH}^*(\mathrm{G}(2,4))$. Prove that $[\sigma_{i,j}] \in \mathrm{CH}^{i+j}$ and for the intersection products one finds $\sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}$ and $\sigma_{1,0}\sigma_{2,1} = [\mathrm{pt}]$.

Exercise 26. *Grothendieck/Hirzebruch–Riemann–Roch formula.*
(i) Use the Euler sequence to compute all Chern classes of $\mathcal{T}_{\mathbb{P}^n}$ and the Hirzebruch–Riemann–Roch formula to compute $\chi(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n})$.
(ii) The *Mukai vector* $v(E)$ of a coherent sheaf E on a smooth variety X is defined as $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(\mathcal{T}_X)}$ (where $\sqrt{\mathrm{td}(\mathcal{T}_X)}$ is defined formally by using that $\mathrm{td} = 1 + \cdots$). Show that the Mukai vector of the structure sheaf of the diagonal $\Delta \subset X \times X$ satisfies $v(\mathcal{O}_\Delta) = [\Delta]$.

Due Wednesday 30 November, 2016.