Dr. Andrey Soldatenkov

Steklov Mathematical Institute

Spring 2022

Introduction to Kähler geometry

Exercise sheet 1

Exercise 1.1. Let I be an almost-complex structure on a manifold M. Prove that I is integrable if and only if for any real vector fields $u, v \in TM$ we have $[u,v] + I[Iu,v] + I[u,Iv] - [Iu,Iv] = 0$.

Exercise 1.2. Let V be a complex vector space of dimension n and $\mathrm{Gr}(m,V)$ the set of m-dimensional subspaces of V for some $m < n$. Describe the natural structure of a complex manifold on $\mathrm{Gr}(m,V)$ and compute its dimension.

Exercise 1.3. Let E be a complex vector bundle over a complex curve C (that is a complex manifold of dimension one), and $\bar\partial_E \colon E \to \Lambda^{0,1}C \otimes E$ be such that $\bar\partial_E(fs) = \bar\partial f \otimes s + f\bar\partial_E s$ for any $f \in \mathcal{O}_C$ and $s \in E$. Prove that for every point $x \in C$ there exists an open neighbourhood U of x and a section s of E over U such that $\bar\partial_E s = 0$ and $s(y) \neq 0$ for any $y \in U$. Deduce that E admits the structure of a holomorphic vector bundle. For simplicity first consider the case $\mathrm{rk}(E) = 1$.

Exercise 1.4. Let E be a holomorphic vector bundle of rank r over a complex manifold M and s a global holomorphic section of E. Assume that the zero locus $Z = \{x \in M \mid s(x) = 0\}$ is a smooth submanifold of codimension r in M. Prove the adjunction formula $K_Z \simeq (K_M \otimes \det(E))|_Z$.

Exercise 1.5. For the exact triple of vector bundles

$$0 \to L \to E \to Q \to 0,$$

where $\mathrm{rk}(L) = 1$, construct the exact triple

$$0 \to L \otimes \Lambda^{k-1}Q \to \Lambda^k E \to \Lambda^k Q \to 0.$$

Exercise 1.6.

1. Compute $\Gamma(\mathbb{C}P^n, \mathcal{O}(k))$;
2. Find the canonical bundle of $\mathbb{C}P^n$.

Exercise 1.7. A complex manifold is Stein if for some n it can be embedded into $\mathbb{C}^n$ as a closed submanifold. Let M be an arbitrary complex manifold and $U, V \subset M$ two open subsets such that both U and V are Stein manifolds. Prove that $U \cap V$ is also Stein.