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Introduction to Kähler geometry

Exercise sheet 2

Exercise 2.1. Let (M, I) be a complex manifold and X a real vector field on M. Prove that $X^{1,0}$ is a holomorphic vector field if and only if $L_X I = 0$.

Exercise 2.2. Prove that

$$\omega = \sqrt{-1}\partial\overline{\partial}\log\left(1 + \sum_{i=1}^{n} |x_i|^2\right)$$

is a Kähler form on $\mathbb{C}^n$. Prove that this form is the restriction of the Fubini-Study form on $\mathbb{C}P^n$.

Exercise 2.3. Prove that every Hermitian metric on a complex curve is Kähler.

Exercise 2.4. Prove that $\mathbb{C}P^1$ with the Fubini-Study metric is isometric to a round sphere in the Euclidean space $\mathbb{R}^3$ with the induced metric.

Exercise 2.5. Let ω be a Kähler form on a complex manifold M and $[\omega] \in H^2(M, \mathbb{R})$ its cohomology class, where we use the isomorphism between singular and de Rham cohomologies of M.

1. Using the local expression for ω prove that ω^m is a volume form, where $m = \dim_{\mathbb{C}} M$;
2. Prove that $[\omega] \neq 0$ if M is compact;
3. Consider the Hopf manifold $M = (\mathbb{C}^m \setminus \{0\})/\mathbb{Z}$, where $\mathbb{Z}$ acts by dilations $z \mapsto \lambda^k z$ for some $\lambda > 1$. Prove that for $m > 1$ the manifold M does not admit any Kähler metric.

Exercise 2.6. Let M be a connected Kähler manifold with Kähler metric g. Prove that g is the unique Kähler metric in its conformal class, i.e. if for some $f \in C^\infty_M$ the metric $e^f g$ is Kähler, then f is constant.

Exercise 2.7. Let M be a complex manifold and $N \subset M$ a compact complex submanifold, $\dim_{\mathbb{C}} N = k$. Observe that any complex manifold has a natural orientation defined by its complex structure, therefore it has a well-defined fundamental class. Denote by $[N] \in H_{2k}(M, \mathbb{Z})$ the homology class of N, i.e. the direct image of the fundamental class of N in the homology of M.

1. Prove that $[N] \neq 0$ if M is Kähler;
2. Construct an example of N inside the Hopf manifold (see exercise 2.5.3) such that $[N] = 0$.