

Introduction to the Moduli Spaces of Sheaves on K3 Surfaces

Exercise sheet 1

Exercise 1.1. Let C be a smooth curve and $F \in \text{Coh}(C)$. Prove that $F \simeq T \oplus E$ where T is a torsion sheaf and E is a vector bundle.

Exercise 1.2. Prove that any vector bundle E over \mathbb{P}^1 is isomorphic to $\bigoplus_{i=1}^{\text{rk}(E)} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some $a_i \in \mathbb{Z}$ as follows:

1. Prove that there exists $k_0 \in \mathbb{Z}$ such that $H^0(\mathbb{P}^1, E(k_0)) \neq 0$ but $H^0(\mathbb{P}^1, E(k)) = 0$ for $k < k_0$;
2. Construct a short exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow E(k_0) \rightarrow E' \rightarrow 0$ and prove that E' is torsion-free;
3. Arguing by induction on the rank of E , conclude that $E(k_0)$ has the required form.

Exercise 1.3. Give an example of a vector bundle E over \mathbb{P}^2 such that $H^0(\mathbb{P}^2, E) = 0$ but $H^0(L, E|_L) \neq 0$ for any line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$.

Exercise 1.4*. Assume that E is a vector bundle over \mathbb{P}^2 such that $E|_L$ is trivial for any line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$. Prove that E is trivial.

Exercise 1.5. Assume that X is a projective manifold and $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is a short exact sequence of coherent sheaves over X . Is it always possible to construct $\mathcal{G} \in \text{Coh}(X \times \mathbb{A}^1)$ such that \mathcal{G} is flat over \mathbb{A}^1 , $\mathcal{G}_0 \simeq F$ and $\mathcal{G}_t \simeq F' \oplus F''$ for $t \neq 0$, where \mathcal{G}_t is the fibre of \mathcal{G} over the point $t \in \mathbb{A}^1$?

Exercise 1.6. Give an example of a coherent sheaf $\mathcal{F} \in \text{Coh}(X \times \mathbb{A}^1)$ with the following property: $H^0(X, \mathcal{F}_t) \neq 0$ for $t \neq 0$ and $H^0(X, \mathcal{F}_0) = 0$. *Hint:* \mathcal{F} should not be flat over \mathbb{A}^1 .