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Introduction to the Moduli Spaces of Sheaves on K3 Surfaces

Exercise sheet 2

Exercise 2.1. Let V be a 3-dimensional complex vector space and $\mathbb{P}(V)$ the corresponding projective space. Recall that the points of the dual space $\mathbb{P}(V^*)$ correspond to lines $\ell \subset \mathbb{P}(V)$. Consider the incidence variety

$$D = \{(x, \ell) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid x \in \ell\}$$

and the two projections $p_1 \colon D \to \mathbb{P}(V)$, $p_2 \colon D \to \mathbb{P}(V^*)$. Lel $L \subset \mathbb{P}(V^*)$ be a line and $Z = p_2^{-1}(L)$ its preimage in D. Describe the fibres of the restricion of p_1 to Z. Is the structure sheaf $\mathcal{O}_Z$ flat over $\mathbb{P}(V)$?

Exercise 2.2*. Assume that $F \in Coh(X \times T)$ is flat over T and for any $t \in T$ the restriction F_t to the fibre over t is locally free. Prove that F is locally free.

Exercise 2.3.** Assume that X and T are two varieties and $f \colon X \to T$ a flat surjective morphism, i.e. the structure sheaf $\mathcal{O}_X$ is flat over T. Assume that X is non-singular. Prove that T is also non-singular.

Exercise 2.4. Let $0 \to F' \to F \to F'' \to 0$ be an exact triple of coherent sheaves. Assume that F' and F'' are pure of dimension d. Prove that F is also pure of dimension d.

Exercise 2.5. Is the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(10) \oplus \mathcal{O} \oplus \mathcal{O}$ on $\mathbb{P}^1$ (semi-)stable? If not, write down the Harder–Narasimhan filtration for this bundle.

Exercise 2.6. Compute the Hilbert polynomial of the ideal sheaf of n points on $\mathbb{P}^2$.

Exercise 2.7. Let $f \colon E \to F$ be a morphism of stable sheaves. Let $P_{\mathrm{red}}(E)$ and $P_{\mathrm{red}}(F)$ be the reduced Hilbert polynomials of E and F.

1. Assume that $P_{\mathrm{red}}(E) > P_{\mathrm{red}}(F)$. Prove that $f = 0$.
2. Assume that $P_{\mathrm{red}}(E) = P_{\mathrm{red}}(F)$. Prove that f is either zero or an isomorphism.

Exercise 2.8. Prove that the tangent bundle of $\mathbb{P}^3$ is slope stable.