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Introduction to the Moduli Spaces of Sheaves on K3 Surfaces

Exercise sheet 3

Exercise 3.1. Let $F \in Coh(X)$ be pure of dimension d. Let $F_1 \subset F_2 \subset F$ be two subsheaves.

1. Assume that F_1 is saturated in F_2 and F_2 is saturated in F. Prove that F_1 is saturated in F;
2. Assume that F_1 is saturated in F. Prove that F_1 is saturated in F_2.

Exercise 3.2. Let $\mathcal{O}_X(1)$ be a very ample line bundle on a smooth projective manifold X and F a coherent sheaf on X which is pure of dimension d. Denote $F(1) = F \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$. Prove that F is (semi-)stable if and only if $F(1)$ is (semi-)stable. Assuming that F is torsion-free, compare the slopes of F and $F(1)$.

Exercise 3.3. Assume that $F \in Coh(X)$ is pure of dimension d and $F^+ \subset F$ is the maximal destabilizing subsheaf. Let $A\colon F \to F$ be an endomorphism of F. Prove that A preserves F^+, i.e. $A(F^+) \subset F^+$.

Exercise 3.4. Let X be a projective manifold and $j\colon Y \hookrightarrow X$ a submanifold. Let $\mathcal{O}_Y(1) = j^*\mathcal{O}_X(1)$. Prove that a sheaf $F \in Coh(Y)$ is (semi-)stable as a sheaf on Y if and only if $j_*F \in Coh(X)$ is a (semi-)stable sheaf on X.

Exercise 3.5. Consider a coherent sheaf $E \in Coh(X)$. Let $\theta\colon E \to E \otimes \Omega^1_X$ be a morphism of $\mathcal{O}_X$-modules. We extend θ to a morphism $E \otimes \Omega^k_X \to E \otimes \Omega^{k+1}_X$ for arbitrary $k \geqslant 0$ as follows: we compose $\theta \otimes \mathrm{id}\colon E \otimes \Omega^k_X \to E \otimes \Omega^1_X \otimes \Omega^k_X$ with the antisymmetrization map $\Omega^1_X \otimes \Omega^k_X \to \Omega^{k+1}_X$. The morphism θ is called *a Higgs field* if $\theta^2\colon E \to E \otimes \Omega^2_X$ vanishes. In this case the pair (E, θ) is called *a Higgs sheaf*. A Higgs subsheaf of E is a coherent subsheaf $F \subset E$ such that $\theta(F) \subset F \otimes \Omega^1_X$.

1. Let $E = \mathcal{O}_X \oplus \Omega^1_X$. Define $\theta\colon E \to E \otimes \Omega^1_X$ as the composition of the projection $\mathcal{O}_X \oplus \Omega^1_X \to \Omega^1_X$ and the inclusion $\Omega^1_X \hookrightarrow \Omega^1_X \oplus (\Omega^1_X)^{\otimes 2} \simeq E \otimes \Omega^1_X$. Prove that θ is a Higgs field.
2. Assume that the degree of K_X is positive and the bundle Ω^1_X is slope semistable. Prove that the Higgs bundle (E, θ) defined above is slope stable in the following sense: for any Higgs subsheaf $F \subsetneq E$ we have $\mu(F) < \mu(E)$.