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Introduction to the Moduli Spaces of Sheaves on K3 Surfaces

Exercise sheet 4

Exercise 4.1. Let X be a scheme over $\mathbb{C}$ and $P = \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ (the scheme P is sometimes called "the fat point"). Consider a coherent sheaf $\mathcal{F} \in Coh(X \times_{\mathbb{C}} P)$ and denote by F its restriction to the fibre over $\mathrm{Spec}(\mathbb{C}) \subset P$, i.e. $F = \mathcal{F}/\varepsilon\mathcal{F}$. Prove that $\mathcal{F}$ is flat over P if and only if the natural map $F \to \varepsilon\mathcal{F}$ is an isomorphism of sheaves of $\mathcal{O}_X$-modules.

Exercise 4.2. Consider the zero locus S of a generic section of the following vector bundles E over the varieties X, and prove that S is a K3 surface:

- $X = \mathbb{P}^4$, $E = \mathcal{O}(2) \oplus \mathcal{O}(3)$;
- $X = \mathbb{P}^5$, $E = \mathcal{O}(2)^{\oplus 3}$;
- $X = \mathbb{P}^1 \times \mathbb{P}^2$, $E = \mathcal{O}(2, 3)$;
- $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $E = \mathcal{O}(2, 2, 2)$;
- $X = \mathrm{Gr}(2, 6)$, $E = \mathcal{O}(1)^{\oplus 6}$.

Exercise 4.3. Let $S \to \mathbb{P}^2$ be a double covering branched over a smooth curve $D \subset \mathbb{P}^2$ of degree 6. Prove that S is a K3 surface.

Exercise 4.4. Let S be a $K3$ surface and $f\colon S \to C$ a dominant morphism onto a smooth curve C. Prove that $C \simeq \mathbb{P}^1$. What are the fibres of such a morphism?

Exercise 4.5*. For a projective K3 surface X let $T(X)$ be the minimal sub Hodge structure of $H^2(X, \mathbb{Q})$ that contains $H^{2,0}(X)$. Prove that $T(X)$ is the orthogonal complement of $NS(X)$. Let $\alpha\colon T(X) \to T(X)$ be a Hodge isometry. Prove that $\alpha^n = \mathrm{id}$ for some n.